I.2 COMPLEX NUMBERS

Objectives: After the completion of this section the student should recall

- the definition of the field of complex numbers
- the operations with complex numbers

Contents:

1. Definition of the Field of Complex Numbers:

Standard form

Exponential form

Trigonometric form

Euler's Formula

2. Operations with Complex Numbers:

De Moivre's Formula and n^{th} root

Properties of absolute value and complex conjugate

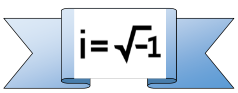
3. Examples

4. Review Questions and Exercises

5. Complex Numbers with Maple

1. DEFINITIONS

The complex numbers is a field of numbers of the form $a+ib$, where a and b are real numbers and i is an imaginary unit. Introduction of the field of the complex numbers is motivated by impossibility of solution of algebraic equation $x^2+1=0$ in the field of real numbers: there is no real $x\in\mathbb{R}$ such that $x^2=-1$. Then the number i was introduced with this property $i^2=-1$. The number i is called the *imaginary unit*. In the field of complex numbers any algebraic equation has a solution.



Definition 1 ***(standard form)***

Complex number z is a number of the form

$$z=a+ib$$

where $a,b\in\mathbb{R}$ are real numbers and

i

is an ***imaginary unit*** of complex number defined as $i=\sqrt{-1}$ or by the property $i^2=-1$. The notations $a+ib$ and $a+bi$ are equivalent.

a is called the ***real part***, $a=Re(z)$ and

b is called the ***imaginary part***, $b=Im(z)$, of the number z

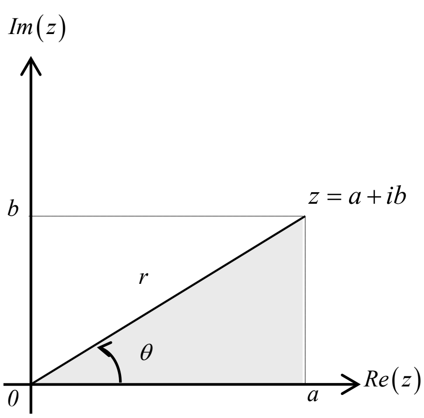
The complex numbers with zero real part, $z=ib$, are called ***pure imaginary numbers***. The complex numbers with zero imaginary part are just real numbers $a+i\cdot 0=a\in\mathbb{R}$.

Addition of complex numbers z_1,z_2 is defined by

$$z_1+z_2 \;=\; (a_1+ib_1)+(a_2+ib_2) \;=\; (a_1+a_2)+i(b_1+b_2)$$

and ***multiplication*** of complex numbers is defined by

$$z_1z_2 \;=\; (a_1+ib_1)(a_2+ib_2) \;=\; (a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)$$



The set of all complex numbers denoted by

$$\mathbb{C}=\left\{a+ib \;\middle|\; a,b\in\mathbb{R},\ i^2=-1\right\}$$

with operations of addition and multiplication of complex numbers defined above is a ***field***. Its zero-element is $0+i0$ and its unit-element is $1+i0$ (multiplicative identity).

Geometrical interpretation of complex numbers is performed with the help of points in the Euclidian plane. If the Cartesian coordinates are associated with the real and the imaginary parts of the complex numbers then the points in the plane with coordinates (a,b) are put in one-to-one correspondence with the complex numbers $z=a+ib$. This geometrical analogy yields two other forms of definition of complex numbers:

Definition 2 ***(vector form)***

The complex numbers are defined as the set of vectors

$$\mathbb{C}=\left\{(a,b) \;\middle|\; a,b\in\mathbb{R}\right\}$$

with operations of addition and multiplication:

$$z_1+z_2 \;=\; (a_1,b_1)+(a_2,b_2) \;=\; (a_1+a_2,b_1+b_2)$$

$$z_1z_2 \;=\; (a_1,b_1)(a_2,b_2) \;=\; (a_1a_2-b_1b_2,a_1b_2+a_2b_1)$$

The zero element is $(0,0)$, the unit-element is $(1,0)$. The vector $(0,1)$ corresponds to the imaginary unit $i=(0,1)$, because it has the same property

$$i^2=(0,1)^2=(0,1)(0,1)=(-1,0)=-1$$

The points in the Euclidean plane also are uniquely defined by the polar coordinates, therefore, there are the other possible forms of definition of complex numbers:

Definition 3 ***(polar form or exponential form, trigonometric form)***

The set of complex numbers is defined as the set

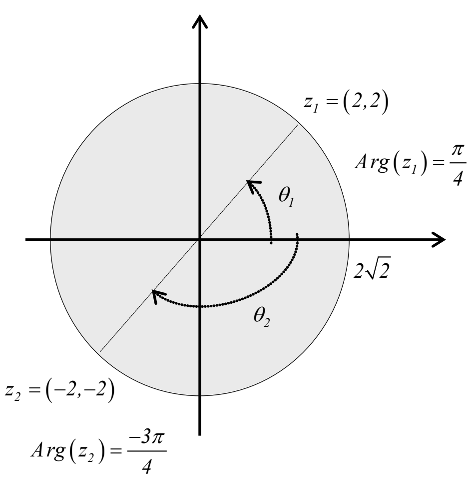
$$\mathbb{C}=\left\{z=a+ib=re^{i\theta} \;\middle|\; a,b\in\mathbb{R}\right\}$$

where $r^2=a^2+b^2$ and $tan\,\theta=\dfrac{b}{a}$. The number

$$r=|z|=\sqrt{a^2+b^2}$$

is called the ***absolute value*** (or ***modulus***) of the complex number z. The geometrical sense of the absolute value is the length of the vector (a,b).

The number θ is called the ***angle*** or the ***argument*** of the complex number z, $\theta=arg(z)$. An argument $arg(z)$ is not unique since $tan\,\theta$ is a periodic function. The value of the argument which is $-\pi<\theta\le\pi$ is called the ***principle argument*** and is denoted by the symbol $Arg(z)$. Because $tan\,\theta$ is π-periodic, the principle argument should be consistent with the quadrant where the point (a,b) is located.



Example: Rewrite two complex numbers $z_1=2+2i$ and $z_2=-2-2i$ in exponential form:

$$|z_1|=|z_2|=\sqrt{2^2+2^2}=2\sqrt{2}$$

$$tan\,\theta_1=\frac{2}{2}=1 \quad\Rightarrow\quad arg(z_1)=\frac{\pi}{4}\pm n\pi\,,\qquad Arg(z_1)=\frac{\pi}{4}$$

$$z_1=(2,2)=2\sqrt{2}e^{\frac{i\pi}{4}}\in I^{st}\,quadrant$$

$$tan\,\theta_2=\frac{-2}{-2}=1 \quad\Rightarrow\quad arg(z_2)=\frac{\pi}{4}\pm n\pi\,,\qquad Arg(z_2)=\frac{-3\pi}{4}$$

$$z_2=(-2,-2)=2\sqrt{2}e^{\frac{-i3\pi}{4}}\in III^{rd}\,quadrant$$

An exponential function of imaginary variable is calculated according to Euler's Formula:

Euler's Formula

$$e^{i\theta}=cos(\theta)+i\,sin(\theta)$$

The Euler's Formula yields also the ***trigonometric form*** of complex numbers:

$$z=a+ib=re^{i\theta}=r\left[cos(\theta)+i\,sin(\theta)\right]$$

2. OPERATIONS:

Let us summarize operations for all introduced forms of complex numbers. Let $k \in \mathbb{R}$ and $z, z_1, z_2 \in \mathbb{C}$ with the following representation forms:

$$z = a + ib = (a,b) = re^{i\theta} = r\left[cos(\theta) + i\,sin(\theta)\right]$$
$$z_1 = a_1 + ib_1 = (a_1, b_1) = r_1 e^{i\theta_1} = r_1\left[cos(\theta_1) + i\,sin(\theta_1)\right]$$
$$z_2 = a_2 + ib_2 = (a_2, b_2) = r_2 e^{i\theta_2} = r_2\left[cos(\theta_2) + i\,sin(\theta_2)\right]$$

Then the following operations with complex numbers are defined (see The Table for geometrical illustration):

Equality

Complex numbers z_1 and z_2 are ***equal*** if their real and imaginary parts are equal:

$z_1 = z_2$ if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$
$a_1 = a_2$ and $b_1 = b_2$
$r_1 = r_2$ and $\theta_1 = \theta_2$

Addition

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + i(b_1 + b_2) \\ &= (a_1 + a_2, b_1 + b_2) \\ &= r_1 e^{i\theta_1} + r_2 e^{i\theta_2} \\ &= r_1 cos(\theta_1) + r_2 cos(\theta_2) + i\left[r_1 sin(\theta_1) + r_2 sin(\theta_2)\right] \end{aligned}$$

Multiplication

$$\begin{aligned} z_1 z_2 &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \\ &= (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1) \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ &= r_1 r_2\left[cos(\theta_1 + \theta_2) + i\,sin(\theta_1 + \theta_2)\right] \end{aligned}$$

Multiplication by a scalar

$$\begin{aligned} kz &= ka + i(kb) \\ &= (ka, kb) \\ &= kre^{i\theta} \\ &= kr\left[cos(\theta) + i\,sin(\theta)\right] \end{aligned}$$

Division

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \qquad \textit{provided } a_2^2 + b_2^2 \neq 0 \\ &= \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}, \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right) \\ &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \\ &= \frac{r_1}{r_2}\left[cos(\theta_1 - \theta_2) + i\,sin(\theta_1 - \theta_2)\right] \end{aligned}$$

Conjugate

$$\begin{aligned} \overline{z} &= a - ib \\ &= (a, -b) \\ &= re^{-i\theta} \\ &= r\left[cos(\theta) - i\,sin(\theta)\right] \end{aligned}$$

Conjugate of the complex number z is obtained by changing the sign of its imaginary part. Geometrically it results in the reflection of the point $z = (a,b)$ over the *x-axis* to $\overline{z} = (a, -b)$.

Powers (De Moivres's Formula)

$$z^n = r^n e^{in\theta} = r^n\left[\cos(n\theta) + i\sin(n\theta)\right] \qquad n \in \mathbb{N}$$

n^{th} Roots

There are $n \in \mathbb{N}$ roots of the equation $x^n = z$ which are all distinct and for $k = 0,1,2,\ldots,n-1$ are given by

$$x_k = \left(z^{\frac{1}{n}}\right)_k = r^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$$

All roots are evenly distributed on the circle with radius $r^{1/n}$.
If $z \in \mathbb{R}$ is a real number, $Im(z) = 0$, then the complex roots appear in conjugate pairs.

Properties of absolute value and complex conjugate:

1) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$

2) $\overline{z_1\, z_2} = \overline{z}_1\, \overline{z}_2$

3) $\overline{k\, z} = k\, \overline{z} \qquad k \in \mathbb{R}$

4) $|z| = 0$ *if and only if* $z = 0$

5) $|z_1 z_2| = |z_1||z_2|$

6) $|z_1 + z_2| \le |z_1| + |z_2|$ *(triangle inequality)*

$|z_1 + z_2| = |z_1| + |z_2|$ *only if* $z_1 = z_2$

7) $|z| = \sqrt{z\overline{z}}$

8) $z^{-1} \equiv \frac{1}{z} = \frac{1+i0}{z} = \frac{\overline{z}}{|z|^2} = \frac{a-ib}{a^2+b^2}$

9) $Re(z) = \frac{1}{2}(z + \overline{z})$

10) $Im(z) = \frac{1}{2i}(z - \overline{z})$

3. EXAMPLES:

1. Derive the multiplication formula for complex numbers in standard form using algebraic rules and property of the imaginary unit $i^2 = -1$:

$$\begin{aligned} z_1 z_2 &= (a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 + i a_1 b_2 + i a_2 b_1 + i^2 b_1 b_2 \\ &= a_1 a_2 + i(a_1 b_2 + a_2 b_1) - b_1 b_2 \\ &= a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1) \end{aligned}$$

2. Derive the multiplication formula for complex numbers in polar and trigonometric forms using rules for exponents, the Euler's formula and trigonometric addition theorem:

First, show that

$$\begin{aligned} e^{i(\theta_1+\theta_2)} &= cos(\theta_1+\theta_2) + i\,sin(\theta_1+\theta_2) \\ &= cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) + i\left[sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2)\right] \\ &= \left[cos(\theta_1) + i\,sin(\theta_1)\right]\left[cos(\theta_2) + i\,sin(\theta_2)\right] \\ &= e^{i\theta_1} e^{i\theta_2} \end{aligned}$$

Then

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2\left[cos(\theta_1+\theta_2) + i\,sin(\theta_1+\theta_2)\right]$$

3. Derive the quotient rule:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1+ib_1}{a_2+ib_2} = \frac{a_1+ib_1}{a_2+ib_2}\frac{a_2-ib_2}{a_2-ib_2} \\ &= \frac{(a_1+ib_1)(a_2-ib_2)}{a_2^2+b_2^2} \\ &= \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2} + i\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2} \end{aligned}$$

4. Calculate $\dfrac{2+3i}{3-4i} = \dfrac{2\cdot 3+3\cdot(-4)}{3^2+4^2} + \dfrac{3\cdot 3-2\cdot 4}{3^2+4^2}i = \dfrac{-6}{25} + \dfrac{17}{25}i$

5. Rewrite the complex numbers in polar or trigonometric form, then perform the indicated operations and return the result to standard form $a+ib$:

$$\begin{aligned} \left(1+\sqrt{3}i\right)\left(\sqrt{3}-i\right) &= \left(2e^{\frac{\pi}{3}i}\right)\left(2e^{\frac{-\pi}{6}i}\right) \\ &= 4e^{\left(\frac{\pi}{3}-\frac{\pi}{6}\right)i} \\ &= 4e^{\frac{\pi}{6}i} \\ &= 2\sqrt{3}+2i \end{aligned}$$

6. Evaluate the complex roots $1^{\frac{1}{3}}$.

Rewrite in trigonometric form:

$$1 = 1 + 0 \cdot i$$

$$= \cos 0 + (\sin 0) i \qquad \theta = 0 \qquad r = 1$$

$$x_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \qquad k = 0,1,2 \qquad n = 3$$

$$= (1)^{\frac{1}{3}}\left[\cos\left(\frac{2k\pi}{3}\right) + i \sin\left(\frac{2k\pi}{3}\right)\right]$$

$$x_0 = 1 \cdot \left[\cos(0) + i \sin(0)\right] \qquad = 1 \qquad (\textit{principle root})$$

$$x_1 = 1 \cdot \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right] \qquad = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$x_1 = 1 \cdot \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right] \qquad = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

These roots are equivalent to the roots of the algebraic equation:

$$x^3 - 1 = (x - 1)\left(x^2 + x + 1\right) = 0$$

7. Evaluate the complex roots $(1+i)^{\frac{1}{5}}$.

Rewrite in trigonometric form:

$$1 + i = \cos\frac{\pi}{4} + \left(\sin\frac{\pi}{4}\right) i \qquad \theta = \frac{\pi}{4} \qquad r = \sqrt{2}$$

$$x_k = \left(\sqrt{2}\right)^{\frac{1}{5}}\left[\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{5}\right) + i \sin\left(\frac{\frac{\pi}{4} + 2k\pi}{5}\right)\right] \qquad k = 0,1,2,3,4 \qquad n = 5$$

$$= \sqrt[10]{2}\left[\cos\left(\frac{\pi + 8k\pi}{20}\right) + i \sin\left(\frac{\pi + 8k\pi}{20}\right)\right]$$

$$x_0 = \sqrt[10]{2}\left[\cos\left(\frac{\pi}{20}\right) + i \sin\left(\frac{\pi}{20}\right)\right] \qquad \approx 1.06 - 0.17i$$

$$x_1 = \sqrt[10]{2}\left[\cos\left(\frac{9\pi}{20}\right) + i \sin\left(\frac{9\pi}{20}\right)\right] \qquad \approx 0.17 + 1.06i$$

$$x_3 = \sqrt[10]{2}\left[\cos\left(\frac{17\pi}{20}\right) + i \sin\left(\frac{17\pi}{20}\right)\right] \qquad \approx -0.96 + 0.49i$$

$$x_3 = \sqrt[10]{2}\left[\cos\left(\frac{25\pi}{20}\right) + i \sin\left(\frac{25\pi}{20}\right)\right] \qquad \approx -0.76 - 0.76i$$

$$x_4 = \sqrt[10]{2}\left[\cos\left(\frac{33\pi}{20}\right) + i \sin\left(\frac{33\pi}{20}\right)\right] \qquad \approx 0.49 - 0.96i$$

4. REVIEW QUESTIONS:

1) What are complex numbers?
2) In what forms can the complex numbers be written?
3) What is the geometrical interpretation of complex numbers?
4) What are the absolute value and the argument of a complex number?
5) What is Euler's formula?
6) Recall the definition of operations with complex numbers.
7) What is the geometrical interpretation of a complex conjugate?
8) Show that $z\bar{z}$ is a real number.

EXERCISES:

1) Write the following expressions in standard form $a+ib$:

a) $2\left(3-\frac{i}{2}\right)$ b) $3(2+i)$

c) $i(3+2i)-2(1-4i)$ d) $i(2-2i)-i(5+9i)$

e) $i(1-2i)(4-4i)$ f) $(1+2i)\left(3+\frac{1}{2}i\right)$

g) $\frac{2+5i}{1-3i}$ h) $\frac{4-i}{(1+3i)(2-i)}$

2) Write the following complex numbers in polar $re^{i\theta}$ and trigonometric form:

a) 4 b) $1+i$

c) $1-i$ d) $-\sqrt{3}-i$

3) Rewrite the complex numbers in polar or trigonometric form, then perform the indicated operations and return the result to standard form $a+ib$:

a) $(2+2i)(-1-i)$ b) $(2-2i)(-1+2i)$

c) $\frac{i}{1-i}$ d) $\frac{8}{-4-4i}$

4) Using the De Moivre's Formula, evaluate the following powers:

a) $(-2+2i)^4$ b) $(-1+2i)^5$

c) $\left(1-\sqrt{3}i\right)^6$ d) $(1-i)^8$

5) Find the inverse of the complex number $z^{-1}\equiv\frac{1}{a+ib}$.

6) Find all roots of complex numbers. Sketch the number and the roots.

a) $(32)^{1/5}$ b) $(-1+i)^{1/3}$

c) $(4+3i)^{1/2}$ d) $\left(1+\sqrt{3}i\right)^{1/6}$

7) Find all roots of the equations if one root is known:

a) $x^5-x^4+x-1=0$ $x=1$

b) $x^5+x^4-16x-16=0$ $x=-1$

COMPLEX NUMBERS WITH MAPLE:

```
> I^2;
```

$$-1$$

```
> I*(2-3*I)*(4+5*I);
```

$$2+23\,I$$

```
> solve(x^4+324=0);
```

$$-3+3\,I,\ -3-3\,I,\ 3+3\,I,\ 3-3\,I$$

```
> solve(x^4+64=0);
```

$$-2+2\,I,\ -2-2\,I,\ 2+2\,I,\ 2-2\,I$$

```
> Im(4-5*I); Re(4-5*I);
```

$$-5$$

$$4$$

```
> abs(4+3*I);
```

$$5$$

```
> argument(2-2*I);
```

$$-\frac{\pi}{4}$$

```
> conjugate (2+3*I);
```

$$2-3\,I$$

```
> polar(1+I);
```

$$\mathrm{polar}\left(\sqrt{2}, \frac{\pi}{4}\right)$$

```
> (2+2*I)*(-1-I);
```

$$-4\,I$$

```
> evalc((2-2*I)*(-1+2*I));
```

$$2+6\,I$$

```
> evalc((1+sqrt(3)*I)/(sqrt(3)-1*I));
```

$$I$$

```
> evalc((-1+2*I)^5);
```

$$-41-38\,I$$

```
> evalf(solve(x^5=1+I));
```

```
1.058578153+.1676623083*I,
.1676623082+1.058578153*I,
-.9549571476+.4865749697*I,
-.7578582833-.7578582831*I,
.4865749698-.9549571476*I
```