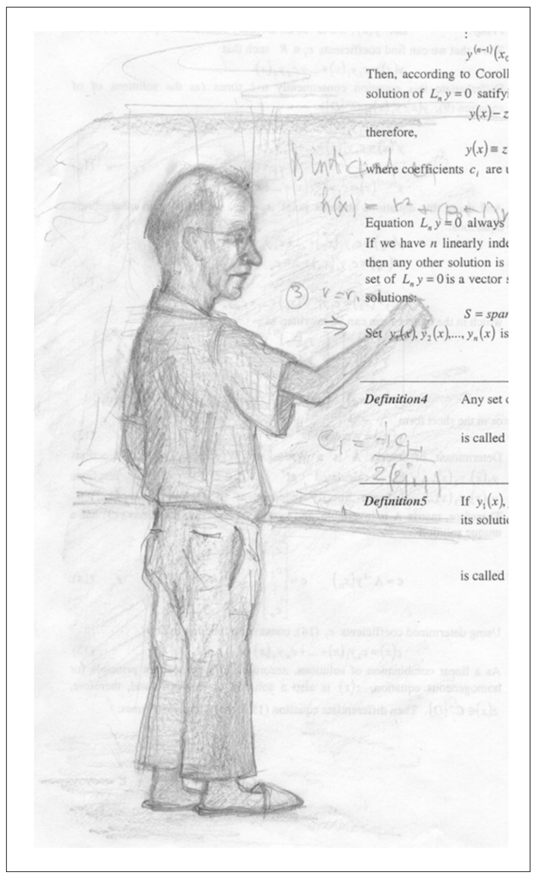
Chapter 5

SPECIAL FUNCTIONS



Chapter 5 Special Functions

Chapter 5

SPECIAL FUNCTIONS

Introduction

In this chapter we summarize information about several functions which are widely used for mathematical modeling in engineering. Some of them play a supplemental role, while the others, such as the Bessel and Legendre functions, are of primary importance. These functions appear as solutions of boundary value problems in physics and engineering.

The survey of special functions presented here is not complete – we focus only on functions which are needed in this class. We study how these functions are defined, their main properties and some applications.

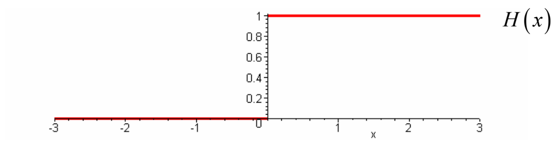
5.1 Heaviside Function (unit step function)

The Heaviside step function has only two values: 0 and 1 with a jump at $x = 0$:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{1}$$

Graphically it can be shown as:

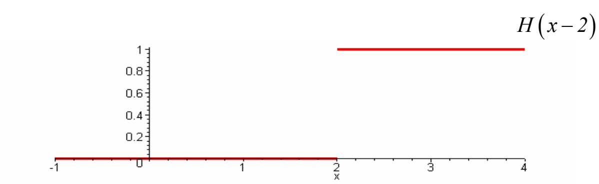
```
> plot(Heaviside(x),x=-3..3,discont=true);
```



Shifting of the step function along the x-axis:

$$H(x-a) = \begin{cases} 0 & x < a \\ 1 & x < a \end{cases} \tag{2}$$

```
> plot(Heaviside(x-2),x=-1..4,discont=true);
```



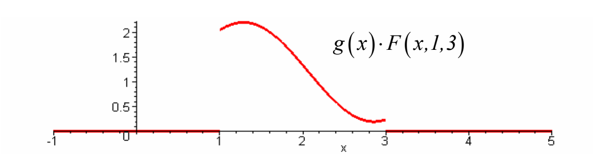
filter function

The **filter function** can be constructed in terms of the step function:

$$F(x,a,b) = H(x-a) - H(x-b) = \begin{cases} 0 & x < a \\ 1 & a < x < b \\ 0 & x > b \end{cases} \tag{3}$$

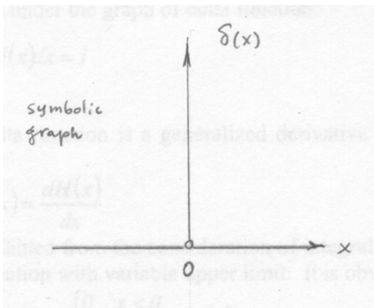
It cuts the values of functions to zero outside of the interval $[a,b]$:

```
> F(x,1,3):=Heaviside(x-1)-Heaviside(x-3);
> plot(g(x)*F(x,1,3),x=-1..5,discont=true);
```



The Heaviside step function is used for the modeling of a sudden increase of some quantity in the system (for example, a unit voltage is suddenly introduced into an electric circuit) – we call this sudden increase a spontaneous source.

5.2 Dirac Function (delta function)



The Dirac delta function $\delta(x)$ is not a function in the traditional sense – it is rather a distribution – a linear operator defined by two properties. The first describes its values to be zero everywhere except at $x = 0$ where the value is infinite:

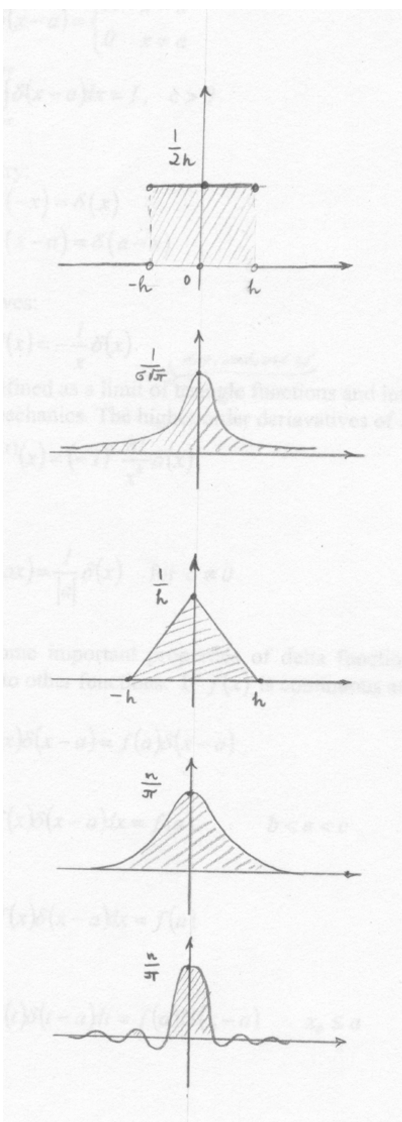
$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{4}$$

The second property provides the unit area under the graph of the delta function:

$$\int_a^b \delta(x)dx = 1 \qquad \text{where } a < 0 \text{ and } b > 0$$

The delta function is vanishingly narrow at $x = 0$ but nevertheless encloses a finite area. It is also known as the unit impulse function.

The Dirac delta function can be treated as the limit of the sequence of the following functions:



a) rectangular functions:

$$\delta(x) = \lim_{h \to 0} S_h(x) = \lim_{h \to 0} \frac{H(x+h) - H(x-h)}{2h}$$

b) Gauss distribution functions:

$$\delta(x) = \lim_{\sigma \to 0} G_\sigma(x) = \lim_{\sigma \to 0} \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}}$$

c) triangle functions:

$$\delta(x) = \lim_{h \to 0} \delta_h(x) \qquad \delta_h(x) = \begin{cases} 0 & x < -h \\ \dfrac{-x}{h^2} + \dfrac{1}{h} & -h \le x < 0 \\ \dfrac{x}{h^2} + \dfrac{1}{h} & 0 \le x \le h \\ 0 & x > h \end{cases}$$

d) Cauchy density (distribution) functions:

$$\delta(x) = \lim_{n \to \infty} D_n = \lim_{n \to \infty} \frac{n}{\pi\left(1 + n^2x^2\right)}$$

e) sine functions:

$$\delta(x) = \lim_{n \to \infty} \frac{\sin nx}{\pi x}$$

Properties

1) Extension of the interval of integration to all real numbers still keeps the unit area under the graph of the delta function:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

2) The Dirac delta function is a generalized derivative of the Heaviside step function:

$$\delta(x) = \frac{dH(x)}{dx}$$

It can be obtained from the consideration of the integral from the definition of the delta function with variable upper limit. It is obvious, that

$$\int_{-\infty}^{x} \delta(t)dt = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} = H(x)$$

Therefore, the step function is formally an antiderivative of the delta function which now can be interpreted as a derivative of a discontinuous function.

3) Shifting in x:

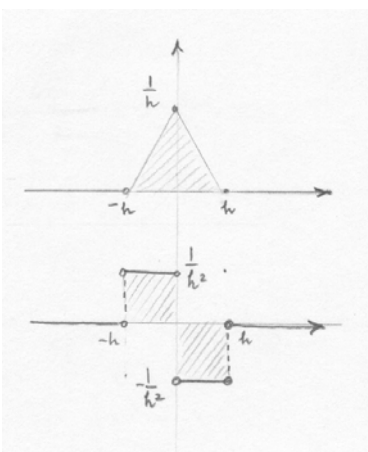
$$\delta(x-a) = \begin{cases} \infty & x = a \\ 0 & x \neq a \end{cases}$$

$$\int_{a-c}^{a+c} \delta(x-a)dx = 1, \quad c > 0$$

4) Symmetry:

$$\delta(-x) = \delta(x)$$

$$\delta(x-a) = \delta(a-x)$$



5) Derivatives:

$$\delta'(x) = -\frac{1}{x}\delta(x)$$

The derivative can be defined as a limit of triangle functions and interpreted as a pure torque in mechanics. The higher order derivatives of the delta function are:

$$\delta^{(k)}(x) = (-1)^k \frac{k!}{x^k}\delta(x) \qquad k = 1, 2, \ldots$$

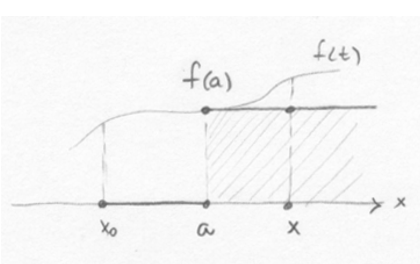
6) Scaling:

$$\delta(ax) = \frac{1}{|a|}\delta(x) \quad \text{for } a \neq 0$$

7) There are some important properties of the delta function which reflect its application to other functions. If $f(x)$ is continuous at $x = a$, then

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

$$\int_{b}^{c} f(x)\delta(x-a)dx = f(a) \qquad b < a < c$$



$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

$$\int_{x_0}^{x} f(t)\delta(t-a)dt = f(a)H(x-a) \qquad x_0 \le a$$

Integration with derivatives of the delta function (integration by parts):

$$\int_{-\infty}^{\infty} f(x)\delta'(x)dx = f(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x)dx = -f'(0)$$

$$\int_{-\infty}^{\infty} f(x)\delta''(x)dx = f(x)\delta'(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta'(x)dx = -\left[-f'(0)\right]' = f''(0)$$

8) Laplace transform:

$$L\{\delta(x)\} = \int_0^{\infty} e^{-sx}\delta(x)dx = 1$$

$$L\{\delta(x-a)\} = \int_0^{\infty} e^{-sx}\delta(x-a)dx = e^{-as} \qquad a>0$$

9) Fourier transform:

$$F\{\delta(x-a)\} = \int_{-\infty}^{\infty} e^{-i\omega x}\delta(x-a)dx = e^{-ia\omega} \qquad a>0$$

Applications

The delta function is applied for modeling of impulse processes.

For example, the unit volumetric heat source applied instantaneously at time $t=0$ is described in the Heat Equation by the delta function:

$$\frac{\partial u}{\partial t} - k\nabla^2 u = \delta(t)$$

If the unit impulse source is located at the point $\mathbf{r} = \mathbf{r}_0$ and releases all energy instantaneously at time $t = t_0$, then the Heat Equation has a source

$$\frac{\partial u}{\partial t} - k\nabla^2 u = \delta(t-t_0)\delta(\mathbf{r}-\mathbf{r}_0)$$

Impulse models are used for calculation of the **Green's function** for non-homogeneous DE.

The other interpretation of the delta function $\delta(t-t_0)$ as a force applied instantaneously at time $t=t_0$ yielding an impulse of unit magnitude.

Example Consider IVP: unit impulse is imposed on a dynamical system initially at rest at $t=5$:

$$y'' + 9y = \delta(t-5)$$

initial conditions: $y(0)=0$

$$y'(0)=0$$

Solution: Apply the Laplace transform to the given initial value problem (use the property of the Laplace transform):

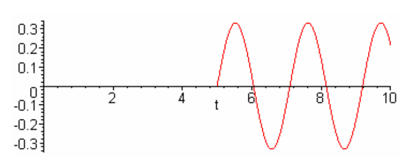
$$s^2Y + 9Y = e^{-5s}$$

Solve the algebraic equation for Y:

$$Y = \frac{e^{-5s}}{s^2+9}$$

The inverse Laplace transform yields a solution of IVP:

$$y(t) = \frac{1}{3}H(t-5)\sin 3(t-5)$$



The graph of the solution shows that the system was at rest until the time $t=5$, when an impulse force was applied yielding periodic oscillations.

5.3 Sine Integral Function

The sine integral function is defined by the formula:

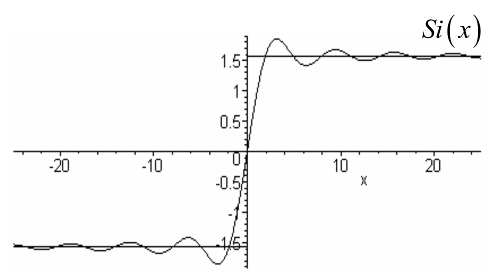
$$\boxed{Si(x) = \int_0^x \frac{\sin t}{t}\,dt} \qquad x \in (-\infty, \infty) \qquad (5)$$

The integrand can be expanded in Taylor series and then integrated term by term yielding a series representation of the sine integral function:

$$Si(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \qquad (6)$$

Graphically it can be shown as:

```
> plot(Si(x),x=-25..25);
```



The limiting values of the sine integral function are determined by the Dirichlet integral

$$\int_0^{\infty} \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2}$$

which was obtained in Section 3.5 (Remark 3.7, p.227) as a particular case of the Fourier transformation of the step function.

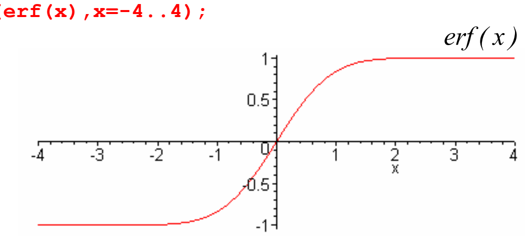
In Chapter 3, we discussed connection of Gibbs phenomena in the Fourier series approximations of functions with jumps and the properties of sine integral function.

5.4 Error Function

The **error function** is the integral of the Gauss density function

$$\boxed{erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt} \qquad x \in (-\infty, \infty) \tag{7}$$

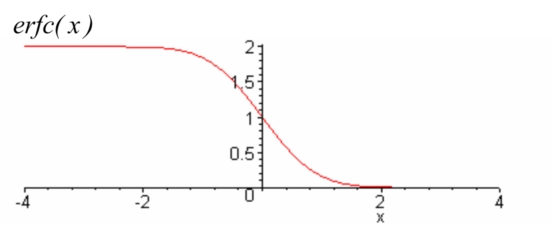
```
> plot(erf(x),x=-4..4);
```



The **complimentary error function** is defined as

$$erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt \tag{8}$$

```
> plot(erfc(x),x=-4..4);
```



Series expansion of the error function:

$$erf(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$

Derivatives of the error function:

$$\frac{d}{dx} erf(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$\frac{d^2}{dx^2} erf(x) = \frac{-4x}{\sqrt{\pi}} e^{-x^2}$$

5.5 Gamma Function

Definition

The Gamma function appears in many integral or series representations of special functions. Gamma was introduced by Leonard Euler in 1729 who investigated the integral function

$$\int_0^1 x^p \left(1-x\right)^q \, dx \qquad p,q \in \mathbb{R}$$

which for natural values $p,q \in \mathbb{N}$ is equal to

$$\frac{p!\,q!}{\left(p+q+1\right)!}$$

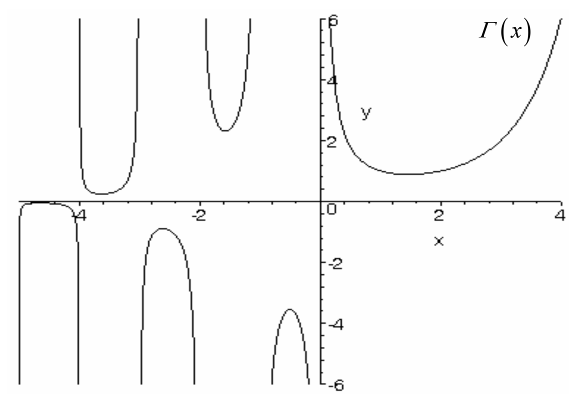
With some transformation of this integral and taking the limits, Euler ended up with the result

$$\int_0^1 \left(-\ln x\right)^n dx = n! \equiv \Gamma\left(n+1\right)$$

Later, the gamma function was defined by the improper integral which converges for all x except of 0 and negative integers (Euler, 1781):

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \tag{9}$$

```
> plot(GAMMA(x),x=-5..4);
```



Properties:

a) $$\boxed{\Gamma(x+1) = x\Gamma(x)} \tag{10}$$

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty e^{-t} t^{(x+1)-1} dt \\
&= \int_0^\infty e^{-t} t^{x} dt \\
&= -\int_0^\infty t^{x} de^{-t} \\
&= -t^x e^{-t}\Big|_0^\infty + \int_0^\infty e^{-t} dt^x \quad \left(\lim_{t\to\infty} t^x e^{-t} = 0\right) \\
&= x\int_0^\infty e^{-t} t^{x-1} dt \\
&= x\Gamma(x)
\end{aligned}$$

b) When $x = n$ is a natural number then

$$\Gamma(n) = (n-1)! \qquad n = 1,2,3,...$$

$$\Gamma(n+1) = n! \qquad n = 0,1,2,... \text{ provided that } 0! = 1 \tag{11}$$

The gamma function is a generalization to real numbers of a factorial (which is defined only for non-negative integers).

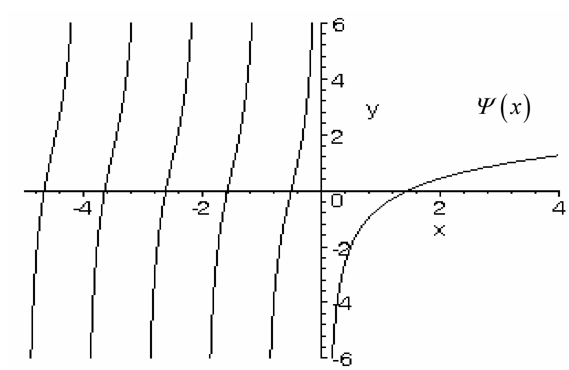
Proof: $\Gamma(1) = 1$ then using property (a)
$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1 = 1!$
$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2 \cdot 1 = 2!$

… (then by mathematical induction)

c) The gamma function does not exist at zero and negative integers.

d) The gamma function is differentiable everywhere except at $x = 0,-1,-2,...$. It is a differentiable extension of the factorial. The derivative of the gamma function is called the digamma function. It is denoted by

$$\Psi(x) = \Gamma'(x)$$



e) Stirling formula (approximation for large x, $x > 9$)

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$$

f) Calculation of gamma function: Lanczos' approximation in Fortran or C++ Numerical recipes.

g) Binomial coefficients:

$$\binom{z}{w} = \frac{z!}{w!(z-w)!} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)} \tag{12}$$

5.6 Bessel Functions

1. Bessel Equation

In the method of separation of variables applied to a PDE in cylindrical coordinates, the equation of the following form appears:

$$r^2 R''(r) + rR'(r) + \left(\lambda^2 r^2 - \nu^2\right)R(r) = 0 \qquad r \in [0,\infty)$$

This equation is the second order linear ordinary differential equation with variable coefficients. It includes two parameters λ and ν. It is not of the Euler-Cauchy type. Can be solved by the **Frobenius method**.

Simplify equation by the change of variables to eliminate parameter λ:

$$y(x) = R(r)$$

$$x = \lambda r$$

$$\frac{dR}{dr} = \frac{dy}{dx}\frac{dx}{dr} = \lambda y'$$

$$\frac{d^2R}{dr^2} = \frac{d}{dr}\left(\frac{dR}{dr}\right) = \frac{d}{dr}\left(\lambda\frac{dy}{dx}\right) = \frac{d}{dx}\left(\lambda\frac{dy}{dx}\right)\frac{dx}{dr} = \lambda^2 y''$$

Then the equation becomes

$$\boxed{x^2 y'' + xy' + \left(x^2 - \nu^2\right)y = 0} \qquad (14)$$

Now the equation is written in traditional variables, and it includes only one parameter ν. This equation is called to be a **Bessel Equation of order** ν. Apply a power-series solution to this equation.

2. Singular Points

Singular points of the differential equation with variable coefficients are the points x at which the first coefficient becomes zero:

$x^2 = 0 \quad \Rightarrow \quad x = 0$ is the only **singular point** of the Bessel Equation. Therefore, if we find a power-series solution around this point, it will be convergent for all real numbers.

Determine the type of the singular point.

Divide the equation by x^2 to rewrite it in the normal form:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

Identify coefficients of the equation in normal form:

$$P(x) = \frac{1}{x} \quad \text{and} \quad Q(x) = 1 - \frac{\nu^2}{x^2}.$$

Check if the singularity is removable:

$xP(x) = 1$ is analytic: $p_0 = 1$

$x^2Q(x) = x^2 - \nu^2$ is analytic: $q_0 = -\nu^2$

Therefore, $x = 0$ is a **regular singular point**, and the Frobenius theorem can be used for solution of the Bessel Equation.

3. Indicial Equation

Substitute coefficients p_0 and q_0 into the indicial equation (Ch. 2.42):

$$r^2 + (p_0 - 1)r + q_0 = 0$$

$$r^2 - \nu^2 = 0$$

There are two roots of this equation:

$$r_1 = \nu \qquad r_2 = -\nu$$

(choose $\nu \ge 0$ for convenience, later we can abandon this assumption).

The Frobenius approach depends on the form of the difference of roots of the indicial equation:

$$r_1 - r_2 = \nu - (-\nu) = 2\nu$$

Two cases of the Frobenius theorem may be involved:

a) $r_1 - r_2 = 2\nu \neq integer$

b) $r_1 - r_2 = 2\nu = integer$

this case includes $\nu = n$, $n \in N_0$ (positive integers and zero) and $\nu = \dfrac{2n+1}{2}$ (half of the odd integer)

In both cases, the first solution, following the Frobenius theorem, has to be found in the form:

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1} = \sum_{n=0}^{\infty} c_n x^{n+\nu} \ , \quad x > 0 \tag{15}$$

Proceed to this solution, and then we will analyze how it handles the abovementioned cases.

4. First Solution

Using assumed form of solution (2), calculate the derivatives

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+\nu}$$

$$y_1' = \sum_{n=0}^{\infty} (n+\nu) c_n x^{n+\nu-1}$$

$$y_1'' = \sum_{n=0}^{\infty} (n+\nu)(n+\nu-1) c_n x^{n+\nu-2}$$

and substitute them into the Bessel Equation (1):

$$\sum_{n=0}^{\infty} (n+\nu)(n+\nu-1) c_n x^{n+\nu} + \sum_{n=0}^{\infty} (n+\nu) c_n x^{n+\nu} + \sum_{n=0}^{\infty} c_n x^{n+\nu+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+\nu} = 0$$

Divide the equation by x^{ν} and collect the terms

$$\sum_{n=0}^{\infty} n(n+2\nu) c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

Rename indices:

in the first sum $m = n$; in the second sum $n = m+2$

$$\sum_{m=0}^{\infty} m(m+2\nu) c_m x^m + \sum_{m=2}^{\infty} c_{m-2} x^m = 0$$

Combine both series:

$$0 \cdot c_0 + (1+2\nu) c_1 x + \sum_{m=2}^{\infty} \left[m(m+2\nu) c_m + c_{m-2} \right] x^m = 0 \tag{16}$$

Applying the Identity Theorem 2.6 to the term with summation, we obtain a recurrence relationship:

$$\boxed{c_m = \frac{-c_{m-2}}{m(m+2\nu)}} \quad \text{for } m \geq 2$$

Using this relationship and also the first two terms of the equation, we get:

$m = 0 \quad 0 \cdot c_0 = 0 \qquad \Rightarrow \quad c_0 = arbitrary$

$m = 1 \quad (1+2\nu) c_1 = 0 \qquad \Rightarrow \quad c_1 = 0$ (by assumption, $\nu > 0$)

$m = 2 \quad c_2 = \dfrac{-c_0}{2(2+2\nu)} = \dfrac{-c_0}{2 \cdot 2(1+\nu)} \qquad \Rightarrow \quad c_2 = \dfrac{-c_0}{2 \cdot 2(1+\nu)}$

$m = 3 \quad c_3 = \dfrac{-c_1}{3(3+2\nu)} = 0 \qquad \Rightarrow \quad c_3 = 0$

$m = 4 \quad c_4 = \dfrac{-c_2}{4(4+2\nu)} = \dfrac{c_0}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2(1+\nu)(2+\nu)} \qquad \Rightarrow \quad c_4 = \dfrac{c_0}{2^4 \cdot 2(1+\nu)(2+\nu)}$

$m=5 \quad c_5 = \frac{-c_3}{5(5+2\nu)} = 0 \qquad \Rightarrow \quad c_5 = 0$

$m=6 \quad c_6 = \frac{-c_4}{6(6+2\nu)} = \frac{-c_4}{2\cdot 2\cdot 3(3+\nu)} \qquad \Rightarrow \quad c_6 = \frac{-c_0}{2^4\cdot 3!(1+\nu)(2+\nu)(3+\nu)}$

…

All coefficients with odd indices are equal to zero. Recognizing the pattern, we can determine the coefficients with even indices:

$$c_{2k} = \frac{(-1)^k c_0}{2^{2k}\cdot k!(1+\nu)(2+\nu)\cdots(k+\nu)} \qquad k = 0,1,2,...$$

This expression may be written in more elegant form if the gamma function is used. Multiply and divide the expression by $\Gamma(1+\nu)$:

$$c_{2k} = \frac{(-1)^k c_0}{2^{2k}\cdot k!}\frac{\Gamma(1+\nu)}{\Gamma(1+\nu)(1+\nu)(2+\nu)\cdots(k+\nu)}$$

Repeatedly using the property (a) of the gamma function, we squeeze the product in the denominator:

$$\Gamma(1+\nu)(1+\nu)(2+\nu)\cdots(k+\nu) = \Gamma(2+\nu)(2+\nu)(3+\nu)\cdots(k+\nu) = \Gamma(k+\nu+1)$$

Then the expression for the coefficients becomes:

$$c_{2k} = \frac{(-1)^k c_0}{2^{2k}\cdot k!}\frac{\Gamma(\nu+1)}{\Gamma(\nu+k+1)}$$

Choose the value for the arbitrary coefficient $c_0 = \frac{1}{2^\nu\,\Gamma(\nu+1)}$, then

$$c_{2k} = \frac{(-1)^k}{2^{2k+\nu}\cdot k!\,\Gamma(\nu+k+1)}$$

Then the solution becomes

$$y_1(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu}\cdot k!\,\Gamma(\nu+k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}$$

This series solution converges absolutely for all $x \ge 0$ because there are no other singular points. The function represented by this power-series solution is called to be a **Bessel function of the 1st kind of order** ν and it is denoted by

Bessel function of the 1st kind

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}} \qquad (17)$$

This formula is valid for any real $\nu \ge 0$ (including integers $\nu = n$ and half of the odd integers $\nu = \frac{2n+1}{2}$).

If ν is integer (let $\nu = n$), then the gamma function is replaced by the factorial

$\Gamma(n+k+1) = (n+k)!$

and the solution simplifies to:

$$\boxed{J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+n}}{k!(n+k)!}} \qquad (18)$$

This is a Bessel function of the 1st kind of integer order (including zero).

5. Second Solution

Case 1) $r_2 = -\nu \neq integer$

(including halfs of the odd integers $\nu = \dfrac{2n+1}{2}$)

Because x appears squared in the equation, the second solution may be obtained from the first by a simple replacement of ν by $-\nu$:

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-\nu}}{k!\,\Gamma(-\nu+k+1)} \tag{19}$$

Functions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. It can be shown that the Wronskian of Bessel functions (17) and (19) is:

$$W[J_\nu(x), J_{-\nu}(x)] = \begin{vmatrix} J_\nu(x) & J_{-\nu}(x) \\ J'_\nu(x) & J'_{-\nu}(x) \end{vmatrix} = -\frac{2\sin\nu\pi}{\pi x} \tag{20}$$

If $\nu \neq n$ is not integer, then the Wronskian is not zero and the Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent.

Then the general solution of the Bessel Equation may be written as

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x) \tag{21}$$

Case 2) $\nu = n$ When ν is integer, the Wronskian (20) is equal to zero for any $x > 0$, therefore, Bessel functions of integer orders $J_n(x)$ and $J_{-n}(x)$ are linearly dependent.

We can show that in this case functions $J_n(x)$ and $J_{-n}(x)$ are just multiples of each other. Indeed, write a Bessel function of negative integer order replacing ν by n in equation (19):

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-n}}{k!\,\Gamma(-n+k+1)} \tag{22}$$

and change the index of summation k by s to make substitution in the exponentiation

$2k - n = 2s + n$

then $k = s + n$, and equation (22) becomes

$$J_{-n}(x) = \sum_{s=-n}^{\infty} \frac{(-1)^{s+n} \left(\frac{x}{2}\right)^{2s+n}}{(s+n)!\,\Gamma(s+1)} \tag{23}$$

Consider a factor in the denominator $\Gamma(s+1)$: when $s+1 \leq 0$ (non-positive integer), the gamma function is unbounded, therefore, the first n terms from $s = -n$ to $s = -1$ in the summation (23) are equal to zero, then taking that into account for integers, $\Gamma(s+1) = s!$, we obtain

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n} \left(\frac{x}{2}\right)^{2s+n}}{(s+n)!\,s!} == (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^{s} \left(\frac{x}{2}\right)^{2s+n}}{(s+n)!\,s!}$$

or

$$J_{-n}(x) = (-1)^n J_n(x) \tag{24}$$

So, function $J_{-n}(x)$ is the function $J_n(x)$ up to the sign.

We need to find the second linearly independent solution.

According to the Frobenius Theorem 2.11, it can be found in the form:

$$J_{-\nu}(x)=\sum_{k=0}^{\infty} d_k x^{k-\nu}+cJ_\nu(x)\ln|x|$$

or we can use the reduction formula (Part I, 2.2.9 3, p.131) to find the second solution:

$$J_{-\nu}(x)=J_\nu(x)\int\frac{dx}{xJ_\nu^2(x)}$$

where long division and the Cauchy product should be used (which is tedious but manageable).

Bessel function of the 2nd kind

But, instead, traditionally, the second independent solution is introduced by the definition of the **Bessel function of the 2nd kind of order** ν :

$$\boxed{Y_\nu(x)=\frac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}} \quad \text{for } \nu\neq n \text{ not integer} \tag{25}$$

and for integers, as the limit

$$\boxed{Y_n(x)=\lim_{\nu\to n}Y_\nu(x)=\frac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}} \tag{26}$$

which appear to exist for all $n=0,\pm1,\pm2,...$ (or $n\in Z$).
Can be derived:

$$Y_n(x)=\frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)J_n(x)-....... \tag{27}$$

$$\gamma=\lim_{m\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}-\ln m\right)=0.5772... \text{ Euler's constant}$$

Functions $Y_n(x)$ have a logarithmic singularity at zero, while functions $J_n(x)$ are finite at zero, that leads to their linear independence. It can be shown that the Wronskian for these functions is given by

$$W[J_\nu(x),Y_\nu(x)]=\begin{vmatrix}J_\nu(x) & Y_\nu(x)\\ J_\nu'(x) & Y_\nu'(x)\end{vmatrix}=\frac{2}{\pi x} \tag{28}$$

General solution of Bessel equation

Functions $J_\nu(x)$ and $Y_\nu(x)$ are linearly independent **<u>for all</u>** ν (including integers), and can be used for construction of the general solution of the Bessel equation:

$$\boxed{y(x)=c_1J_\nu(x)+c_2Y_\nu(x)} \tag{29}$$

When the order of the Bessel equation is **<u>not integer</u>**, the complete solution may be also given only in the terms of Bessel functions of the first kind:

$$\boxed{y(x)=c_1J_\nu(x)+c_2J_{-\nu}(x)} \tag{30}$$

The second solution was derived mostly for integer roots, so, we emphasize it by the following statement: the complete solution of the Bessel equation of **<u>integer order *n*</u>** is given by:

$$\boxed{y(x)=c_1J_n(x)+c_2Y_n(x)} \tag{31}$$

<u>6. Bessel functions of half orders</u>

It is happens that functions of orders $\nu=n\pm\frac{1}{2}$ can be expressed in terms of elementary functions. Show it for $\nu=\pm\frac{1}{2}$. Consider BE (14).

Use substitution $y = \frac{u}{\sqrt{x}} = x^{-\frac{1}{2}} u$

$$y' = -\frac{1}{2} x^{-\frac{3}{2}} u + x^{-\frac{1}{2}} \frac{du}{dx}$$

$$y' = \frac{3}{4} x^{-\frac{5}{2}} u - x^{-\frac{3}{2}} \frac{du}{dx} + x^{-\frac{1}{2}} \frac{du}{dx}$$

then the equation becomes

$$u'' + \left[1 + \frac{\frac{1}{4} - \nu^2}{x^2} \right] u = 0$$

For $\nu = \pm \frac{1}{2}$, this equation reduces to a linear ODE with constant coefficients

$$u'' + u = 0$$

the general solution of which is given by

$$y = c_1 \cos x + c_2 \sin x$$

Apply the back substitution $u = y\sqrt{x}$, then the solution becomes

$$y = c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}}$$

If we choose for constants $c_1 = c_2 = \sqrt{\frac{2}{\pi}}$ to be consistent with definition (17), we obtain that Bessel functions of half orders are given by:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{\sqrt{x}} \tag{32}$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{\sqrt{x}} \tag{33}$$

It can be verified that Bessel functions of $\nu = \pm \frac{3}{2}$ are:

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\frac{\sin x}{x} - \cos x}{\sqrt{x}} \tag{34}$$

$$J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\frac{\cos x}{x} + \sin x}{\sqrt{x}} \tag{35}$$

Other Bessel functions of half of odd integer orders also can be expressed in terms of elementary functions. These functions are used for construction of **spherical Bessel functions**

$$j_n(x) = \sqrt{\frac{\pi}{2}} \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}} = x^n \left(-\frac{1}{x} \frac{d}{dx} \right)^n \frac{\sin x}{x} \tag{36}$$

$$y_n(x) = \sqrt{\frac{\pi}{2}} \frac{Y_{n+\frac{1}{2}}(x)}{\sqrt{x}} = -x^n \left(-\frac{1}{x} \frac{d}{dx} \right)^n \frac{\cos x}{x} \tag{37}$$

which are solutions of the Bessel equation

$$x^2 y'' + 2xy' + \left[x^2 - n(n+1) \right] y = 0 \quad n = 0, \pm 1, \pm 2, ... \tag{38}$$

This equation appears as one of the ODE in the separation of variables of the Laplacian in spherical coordinates [Abramowitz and Stegun, p.437]

7. Bessel functions of the 3rd kind

A linear combination in the general solution (29) assumes that coefficients c_1 and c_2 are real numbers. We also considered the variable x to be a real number too. But the obtained equations and functions are valid also for complex numbers. Two particular combinations of Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ with complex coefficients lead to the introduction of the complex version of Bessel functions, which also are the solutions of the Bessel equation but in the field of complex numbers $z \in Z$:

We define two **Bessel functions of the 3rd kind of order** ν (they are also called **Hankel functions**) as

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) \tag{39}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) \tag{40}$$

or we can express them in terms of the Bessel function $J_\nu(z)$ only if function $Y_\nu(z)$ is replaced in this definition by a complex version of equation (25):

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i \sin \nu\pi} \tag{41}$$

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{\nu\pi i} J_\nu(z)}{-i \sin \nu\pi} \tag{42}$$

Definitions (41) and (42) are for $\nu \neq integer$. For $\nu = n$, $n = 0,1,2,...$, we take the limits:

$$H_n^{(1)}(z) = \lim_{\nu \to n} \frac{J_{-\nu}(x) - e^{-\nu\pi i} J_\nu(x)}{i \sin \nu\pi} \tag{43}$$

$$H_n^{(2)}(z) = \lim_{\nu \to n} \frac{J_{-\nu}(x) - e^{\nu\pi i} J_\nu(x)}{-i \sin \nu\pi} \tag{44}$$

Because Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are linear combinations of Bessel functions of the 1st and the 2nd type, they have the similar properties.

The Wronskian of Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ is

$$W\left[H_\nu^{(1)}(z), H_\nu^{(2)}(z)\right] = -\frac{4i}{\pi z}$$

therefore functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ form a fundamental set for the Bessel equation; and the general solution of the Bessel equation can be written as

$$y(x) = c_1 H_\nu^{(1)}(z) + c_2 H_\nu^{(2)}(z)$$

for any order of the Bessel equation ν (including integers).

8. Properties of Bessel functions

a) Functions $J_\nu(x)$ and $Y_\nu(x)$ are both oscillatory; they have infinitely many roots for $x > 0$.

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147456} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18432} + \cdots$$

$$J_2(x) = \frac{x^2}{8} - \frac{x^4}{96} + \frac{x^6}{3072} - \frac{x^8}{184320} + \cdots$$

Identities

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x) \tag{45}$$

$$J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu+1}(x) \tag{46}$$

Differential identities

$$\frac{d}{dx}\left[x^{\nu} J_\nu(x)\right] = x^{\nu} J_{\nu-1}(x) \qquad \boxed{J_0'(x) = -J_1(x)} \tag{47}$$

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x) \tag{48}$$

$$xJ_\nu'(x) + \nu J_\nu(x) = xJ_{\nu-1}(x) \qquad J_\nu'(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x) \tag{49}$$

$$xJ_\nu'(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x) \qquad J_\nu'(x) = -J_{\nu+1}(x) + \frac{\nu}{x} J_\nu(x) \tag{50}$$

Integral identities

$$\int x^{\nu+1} J_\nu(x)dx = x^{\nu+1} J_{\nu+1}(x) + c \tag{51}$$

$$\int x^{-\nu+1} J_\nu(x)dx = -x^{-\nu+1} J_{\nu-1}(x) + c \tag{52}$$

Addition theorem

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y) \tag{53}$$

The same properties also hold for $Y_\nu(x)$.

b) Functions $I_\nu(x)$ and $K_\nu(x)$ are not oscillatory

$$I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \frac{x^8}{147456} + \cdots$$

$$I_1(x) = \frac{x}{2} + \frac{x^3}{16} + \frac{x^5}{384} + \frac{x^7}{18432} + \cdots$$

$$I_2(x) = \frac{x^2}{8} + \frac{x^4}{96} + \frac{x^6}{3072} + \frac{x^8}{184320} + \cdots$$

9. Generating functions

$$e^{\left(t - \frac{1}{t}\right)\frac{x}{2}} = \sum_{n=-\infty}^{\infty} J_n(x) t^n \tag{55}$$

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - 2J_6(x) + \cdots$$

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) + \cdots$$

(What can be obtained for $x=1$?)

Expansions of x^n

$$1 = J_x(x) + 2J_2(x) + 2J_4(x) + \cdots + 2J_{2k}(x) + \cdots$$

$$\frac{1}{2}x = J_1(x) + 3J_3(x) + 5J_5(x) + \cdots + (2k+1)J_{2k+1}(x) + \cdots$$

$$\frac{1}{8}x = J_2(x) + 4J_4(x) + 9J_6(x) + \cdots + k^2 J_{2k}(x) + \cdots$$

10. Modified Bessel equation

The modified Bessel equation is given by

$$x^2 y'' + xy' - \left(x^2 + \nu^2\right)y = 0 \tag{56}$$

which can be written in the form of Bessel equation (14) with the second parameter $\lambda = i$:

$$x^2 y'' + xy' + \left(i^2 x^2 - \nu^2\right)y = 0 \tag{57}$$

which has a general solution given by equation (29) with x replaced by ix:

$$y(x) = c_1 J_\nu(ix) + c_2 Y_\nu(ix) \tag{58}$$

Equation (58) provides a solution of the modified Bessel equation (91) in complex form. But it is desirable to have a real form of solution. Consider

$$J_\nu(ix) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{ix}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)} = i^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)} \tag{59}$$

because:

$(-1)^k (i)^{2k+\nu} = (-1)^k (i)^{2k} i^\nu = (-1)^k \left(i^2\right)^k i^\nu = (-1)^k (-1)^k i^\nu = (-1)^{2k} i^\nu = i^\nu$.

Then define a function which is called the **modified Bessel function of the 1st kind of order** ν :

$$\boxed{I_\nu(x) = i^{-\nu} J_\nu(ix) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}} \tag{60}$$

which is a real function and which is a solution of the modified Bessel equation (56). Notation I for this function reflects the method of its definition, and it means “the function of imaginary argument”. For negative values of parameter $-\nu$, define a second solution of the modified Bessel equation as

$$\boxed{I_{-\nu}(x) = i^{\nu} J_{-\nu}(ix) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k-\nu}}{k!\,\Gamma(-\nu+k+1)}} \tag{61}$$

The Wronskian of functions $I_\nu(x)$ and $I_{-\nu}(x)$ can be calculated as

$$W\left[I_\nu(x), I_{-\nu}(x)\right] = -\frac{2\sin\nu\pi}{\pi x} \neq 0 \quad \text{for } \nu \neq integer$$

therefore, functions $I_\nu(x)$ and $I_{-\nu}(x)$ are linearly independent and form a fundamental set, then the general solution of the modified Bessel equation of non-integer order is given by

$$\boxed{y(x) = c_1 I_\nu(x) + c_2 I_{-\nu}(x)} \qquad \nu \neq integer \tag{62}$$

In the case of integer orders, function $I_{-n}(x)$ is the same as function $I_n(x)$. Indeed, when ν is changed for n in equation (61)

$$I_{-n}(x) = i^n J_{-n}(ix) = i^n (-1)^n J_n(ix) = i^n (-1)^n i^n \left[i^{-n} J_n(ix)\right]$$
$$= \left(i^2\right)^n (-1)^n I_n(x) = (-1)^n (-1)^n I_n(x) = I_n(x)$$

For integer orders $\nu = n$, $n = 0,1,2,3,\ldots$, the second solution of the modified Bessel equation is defined with the help of the **modified Bessel function of the 2nd kind of order** ν :

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \quad (63)$$

as the limit of

$$K_n(x) = \lim_{\nu \to n} K_\nu(x) \quad (64)$$

11. Equations solvable in terms of Bessel functions

Consider some generalizations of the Bessel equation which also can be solved in terms of the Bessel function.

a) Consider a generalized Bessel differential equation of the form

$$y'' + \left[\frac{1-2m}{x} - 2\alpha\right]y' + \left[p^2 a^2 x^{2p-2} + \alpha^2 + \frac{\alpha(2m-1)}{x} + \frac{m^2 - p^2\nu^2}{x^2}\right]y = 0$$

if $y = y(x)$ is any solution of the Bessel equation, then the function

$$y = x^m e^{\alpha x} y(ax^p)$$

is a solution of the generalized equation. For instance, for any real $\nu \geq 0$ (including integers), the general solution can be written as

$$y = x^m e^{\alpha x}\left[c_1 J_\nu(ax^p) + c_2 Y_\nu(ax^p)\right]$$

or for non-integer orders, a general solution can be written as

$$y = x^m e^{\alpha x}\left[c_1 J_\nu(ax^p) + c_2 J_{-\nu}(ax^p)\right]$$

Proof of this statement can be made by the appropriate change of variables and by reduction of the differential equation to the Bessel equation.

Example 1 Check that the modified Bessel equation

$$x^2 y'' + xy' - (x^2 + \nu^2)y = 0$$

is a particular case of the generalized equation. Rewrite it in the form of the generalized equation:

$$y'' + \left[\frac{1}{x}\right]y' + \left[-1 - \frac{\nu^2}{x^2}\right]y = 0$$

$$y'' + \left[\frac{1-2\cdot 0}{x} - 2\cdot 0\right]y' + \left[(1)(-1)x^{2\cdot 1-2} + \frac{0 - 1\cdot\nu^2}{x^2}\right]y = 0$$

from which we can identify

$m = 0$, $\alpha = 0$, $p = 1$, and $a^2 = -1$

and, therefore, solutions of the modified Bessel equation should include functions

$J_\nu(ix)$, $J_{-\nu}(ix)$, and $Y_\nu(ix)$

what we know from Section 10.

Airy equation

Example 2 Consider the Airy equation

$$y'' - xy = 0$$

which is the simplest case of the linear 2nd order ODE with variable coefficients. This equation has applications in dynamics (oscillation of an aging spring), quantum mechanics and optics.

Rewrite the Airy equation in the form of the generalized equation

$$y'' + \left[\frac{1-2\cdot\frac{1}{2}}{x} - 2\cdot 0\right]y' + \left[\left(\frac{9}{4}\right)\cdot\left(-\frac{4}{9}\right)x^{2\cdot\frac{3}{2}-2} + 0 + 0 + \frac{\frac{1}{4}-\frac{9}{4}\nu^2}{x}\right]y = 0$$

from which we can identify

$m = \frac{1}{2}$, $\alpha = 0$, $p = \frac{3}{2}$, $a^2 = -\frac{4}{9}$, and $\nu^2 = \frac{1}{9}$

from the last equation we can determine the order of the equation

$$\nu = \pm\frac{1}{3}$$

Then solutions of the Airy equation can be written as

$$y_1(x) = x^{\frac{1}{2}} J_{\frac{1}{3}}\left(\frac{2}{3} i x^{\frac{3}{2}}\right)$$

$$y_2(x) = x^{\frac{1}{2}} J_{-\frac{1}{3}}\left(\frac{2}{3} i x^{\frac{3}{2}}\right)$$

If we rewrite Bessel functions of the 1st kind of complex arguments in terms of modified Bessel functions using equation (95), then the solutions become

$$y_1(x) = \sqrt{x}\, I_{\frac{1}{3}}\left(\frac{2}{3} x^{\frac{3}{2}}\right)$$

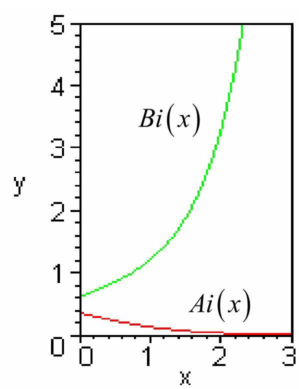
$$y_2(x) = \sqrt{x}\, I_{-\frac{1}{3}}\left(\frac{2}{3} x^{\frac{3}{2}}\right)$$

These two linearly independent solutions (note, that order of modified Bessel functions is not integer) may be used for construction of the traditional form of solutions

$$Ai(x) = \frac{\sqrt{x}}{3}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right]$$

$$Bi(x) = \frac{\sqrt{x}}{3}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right]$$

which are called **Airy functions**. The next plot shows the graph of Airy functions



b) This form of equation is a particular case of the previous equation, but it is more convenient for applications in simpler cases:

$$y'' + a^2 x^{p-2} y = 0$$

solutions of this equation have the forms

$$y(x) = \sqrt{x} J_{\frac{1}{p}}\left(\frac{2a}{p} x^{\frac{p}{2}}\right)$$

$$y(x) = \sqrt{x} Y_{\frac{1}{p}}\left(\frac{2a}{p} x^{\frac{p}{2}}\right)$$

Solutions of the Airy equation can be obtained in this case much faster.

c) Equation

$$y'' + \left(a^2 e^{2x} - p^2\right) y = 0$$

has solutions

$$y(x) = J_p\left(ae^x\right)$$

$$y(x) = J_{-p}\left(ae^x\right)$$

$$y(x) = Y_p\left(ae^x\right)$$

12. Orthogonality of Bessel functions

We know from Sturm-Liouville theory that solutions of the self-adjoint differential equation satisfying homogeneous boundary conditions form a complete set of functions orthogonal with some weight function (Sturm-Liouville theorem). Consider application of this theory to solutions of BVP for BE.

Self-adjoint form of BE

The Bessel equation of order ν with parameter λ

$$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right)y = 0$$

can be reduced to a self-adjoint form with the help of a multiplying factor

$$\mu(x) = \frac{1}{a_0} e^{\int \frac{a_1}{a_0}dx} = \frac{1}{x^2} e^{\int \frac{x}{x^2}dx} = \frac{1}{x^2} e^{\ln x} = \frac{1}{x}$$

After multiplication of BE by $\frac{1}{x}$ it can be reduced to self-adjoint form

$$[xy']' + \left(-\frac{\nu^2}{x} + \lambda^2 x\right)y = 0 \qquad \text{(identify } p(x) = x\text{)}$$

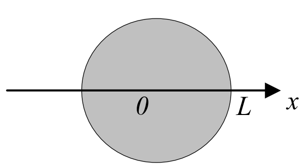
Then, the Sturm-Liouville Problem in the interval $x \in [L_1, L_2]$ produces infinitely many values of the parameter λ_n (eigenvalues) for which there exist non-trivial solutions $y_n(x)$ (eigenfunctions).
According to the Sturm-Liouville theorem, the obtained eigenfunctions are orthogonal with the weight function $p(x) = x$:

$$\int_{L_1}^{L_2} x y_n(x) y_m(x)\,dx = 0 \text{ for } n \neq m$$

Orthogonal sets for circular domain

Consider BE in the finite circle $0 \le x \le L$. The general solution is given by

$0 \le x \le L$



$$y(x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

The physical sense of solution of classical PDE requires a finite value of solution in all points of $[0, L]$. Bessel functions of the second kind $Y_\nu(x)$ are unbounded at $x = 0$, therefore, to satisfy this condition we have to put an arbitrary constant c_2 equal to zero. Then solution of BE becomes

$$y(x) = c_1 J_\nu(\lambda x)$$

boundary conditions:

Consider the homogeneous boundary conditions at $x = L$:

I Dirichlet $y(x)\big|_{x=L} = 0$

II Neumann $y'(x)\big|_{x=L} = 0$

III Robin $\left[y'(x) + Hy(x)\right]_{x=L} = 0$

We are looking for the values of the parameter λ which provide non-trivial solutions $y(x)=c_1 J_\nu(\lambda x)$ satisfying boundary conditions I-III. These values can be found by substitution of the solution into the boundary conditions I-III as the positive roots of the following equations:

Equations for eigenvalues

I $J_\nu(\lambda L)=0$

II $-\lambda J_{\nu+1}(\lambda L)+\frac{\nu}{L}J_\nu(\lambda L)=0$

($\lambda_0=0$ is also an eigenvalue when $\nu=0$)

$$\frac{d}{dx}J_\nu(\lambda x)=0 \qquad \lambda J'_\nu(\lambda x)=0$$

$$\lambda\left[-J_{\nu+1}(\lambda x)+\frac{\nu}{\lambda x}J_\nu(\lambda x)\right]=0$$

III $-\lambda J_{\nu+1}(\lambda L)+\left(H+\frac{\nu}{L}\right)J_\nu(\lambda L)=0$

Proof: $y'(x)+Hy(x)$

$$=\frac{d}{dx}J_\nu(\lambda x)+HJ_\nu(\lambda x) \qquad (=\lambda J'_\nu(\lambda x)+HJ_\nu(\lambda x)\text{, Oz)}$$

$$=\lambda\left[-J_{\nu+1}(\lambda x)+\frac{\nu}{\lambda x}J_\nu(\lambda x)\right]+HJ_\nu(\lambda x)$$

$$=-\lambda J_{\nu+1}(\lambda x)+\left(H+\frac{\nu}{x}\right)J_\nu(\lambda x)$$

In the particular case, when the Bessel function is of zero order, $\nu=0$, equations for eigenvalues are:

I $J_0(\lambda L)=0$

II $J_1(\lambda L)=0$ (and $\lambda_0=0$ is also eigenvalue)

III $-\lambda J_1(\lambda L)+HJ_0(\lambda L)=0$

Orthogonality

Obtained equations generate infinitely many eigenvalues λ_n, $n=1,2,3,...$ For which the corresponding **eigenfunctions** are:

$$\boxed{y_n(x)=\{J_\nu(\lambda_n x)\}}$$

The corresponding set of solutions $\{J_\nu(\lambda_n x)\}$ is orthogonal with respect to the weight function $p(x)=x$:

$$\int_0^L xJ_\nu(\lambda_n x)J_\nu(\lambda_m x)dx=\begin{cases}0 & n\neq m\\ N^2 & n=m\end{cases}$$

where the squared norm of eigenfunctions is determined as (Ozisik N. *Heat Transfer,* p.133; McLachlan *Bessel Functions for Engineers, p.110*):

Norm of eigenfunctions

$$N_{\nu,n}^2 = \int_0^L x J_\nu^2(\lambda_n x)dx = \left[\frac{x^2}{2}\left(J_\nu'^2(\lambda_n x)+\left(1-\frac{\nu^2}{\lambda_n^2 x^2}\right)J_\nu^2(\lambda_n x)\right)\right]_0^L$$

$$= \frac{L^2}{2}\left(J_\nu'^2(\lambda_n L)+\left(1-\frac{\nu^2}{\lambda_n^2 L^2}\right)J_\nu^2(\lambda_n L)\right)$$

or integrating with Maple:

$$N_{\nu,n}^2 = \int_0^L x J_\nu^2(\lambda_n x)dx = \left[\frac{x^2}{2}J_\nu^2(\lambda_n x) - J_{\nu-1}(\lambda_n x)J_{\nu+1}(\lambda_n x)\right]_0^L$$

The derivative can be expressed as (use chain rule and identity in sec. 6)

$$J_\nu'(\lambda_n x) = \frac{d}{dx}J_\nu(\lambda_n x)$$

$$= \lambda_n\left[-J_{\nu+1}(\lambda_n x)+\frac{\nu}{\lambda_n x}J_\nu(\lambda_n x)\right]$$

$$= -\lambda_n J_{\nu+1}(\lambda_n x)+\frac{\nu}{x}J_\nu(\lambda_n x)$$

or if we use the other identity for lower order then

$$J_\nu'(\lambda_n x) = \frac{d}{dx}J_\nu(\lambda_n x)$$

$$= \lambda_n\left[J_{\nu-1}(\lambda_n x)-\frac{\nu}{\lambda_n x}J_\nu(\lambda_n x)\right]$$

$$= \lambda_n J_{\nu-1}(\lambda_n x)-\frac{\nu}{x}J_\nu(\lambda_n x)$$

Then taking into account that eigenvalues λ_n satisfy equations I-III (that simplifies expressions), the squared norm for specific boundary conditions is given:

I $\quad N_{\nu,n}^2 = \frac{L^2}{2}J_{\nu+1}^2(\lambda_n L)$ or $= \frac{L^2}{2}J_{\nu-1}^2(\lambda_n L)$

II $\quad N_{\nu,n}^2 = \frac{L^2}{2}\left(1-\frac{\nu^2}{\lambda_n^2 L^2}\right)J_\nu^2(\lambda_n L) \quad \left(N_{0,0}^2 = \frac{L^2}{2}\right)$

III $\quad N_{\nu,n}^2 = \frac{L^2}{2}\left[\frac{H^2}{\lambda_n^2}+\left(1-\frac{\nu^2}{\lambda_n^2 L^2}\right)\right]J_\nu^2(\lambda_n L)$

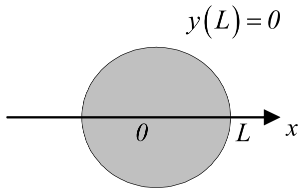
Bessel-Fourier Series

The obtained orthogonal systems can be used for constructing the function expansion in a generalized Fourier series

$$f(x) = \sum_{n=1}^{\infty} a_n J_\nu(\lambda_n x)$$

where coefficients c_n are determined from the equation

$$a_n = \frac{\int_0^L x J_\nu(\lambda_n x) f(x)dx}{\int_0^L x J_\nu^2(\lambda_n x)dx} = \frac{\int_0^L x J_\nu(\lambda_n x) f(x)dx}{N_{\nu,n}^2}$$



Example 3 *(I Dirichlet boundary condition)*

Consider an orthogonal set obtained as a solution of a Dirichlet problem with Bessel functions of zero order $\left\{ y_n^{(\nu)}(x) = J_\nu(\lambda_n x) \right\}$ where eigenvalues λ_n are positive roots of equation $J_\nu(\lambda L) = 0$.
The squared norm of eigenfunctions can be calculated as

$$N_{\nu,n}^2 \quad = \frac{L^2}{2} J_{\nu+1}^2(\lambda_n L)$$

Then expansion of function $f(x)$ in Fourier-Bessel series has the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n x), \quad \text{where } a_n = \frac{2}{L^2} \frac{\int_0^L x J_\nu(\lambda_n x) f(x)\,dx}{J_{\nu+1}^2(\lambda_n L)}$$

(it is also known as the Hankel series (1869)).

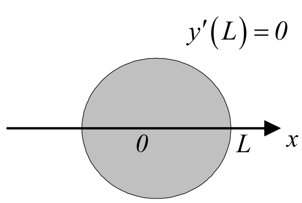
Consider now expansion of the function $f(x) = 1 - H(x-1)$, $x \in [0,3]$ in the Hankel series of order $\nu = 0$. Coefficients are

$$a_n = \frac{\int_0^3 x J_0(\lambda_n x)\,dx}{N_{0,n}^2} = \frac{J_1(\lambda_n)}{\lambda_n} \frac{1}{N_{0,n}^2} = \frac{2}{L^2} \frac{J_1(\lambda_n)}{\lambda_n J_1^2(\lambda_n L)}$$

and the expansion becomes

$$f(x) = \frac{2}{9} \sum_{n=1}^{\infty} \frac{J_1(\lambda_n) J_0(\lambda_n x)}{\lambda_n J_1^2(3\lambda_n)}$$

This example can be illustrated with a Maple presentation (*SF-1.mws*)



Example 4 *(II Neumann boundary condition)*

Consider an orthogonal set obtained as a solution of the Neumann problem with Bessel functions $y_n^{(\nu)}(x) = J_\nu(\lambda_n x)$ where eigenvalues λ_n are positive roots of the equation

$-\lambda J_{\nu+1}(\lambda L) + \dfrac{\nu}{L} J_\nu(\lambda L) = 0$ ($\lambda_0 = 0$ is also an eigenvalue for $\nu = 0$)

The squared norm of eigenfunctions can be calculated as

$$N_{\nu,n}^2 \quad = \frac{L^2}{2}\left(1 - \frac{\nu^2}{\lambda_n^2 L^2}\right) J_\nu^2(\lambda_n L) \text{ and } \left(N_{0,0}^2 = \frac{L^2}{2}\right)$$

Fourier-Bessel series:

$\underline{\nu = 0}$ $\quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n y_n^{(0)}(x)$, $\quad$ where $\quad a_n = \dfrac{\int_0^L x J_0(\lambda_n x) f(x)\,dx}{N_{\nu,n}^2}$

$$a_0 = \frac{2}{L^2} \int_0^L x f(x)\,dx$$

$\underline{\nu > 0}$ $\quad f(x) = \sum_{n=1}^{\infty} a_n y_n^{(\nu)}(x)$, $\quad$ where $\quad a_n = \dfrac{\int_0^L x J_\nu(\lambda_n x) f(x)\,dx}{N_{\nu,n}^2}$

Consider now expansion of the function $f(x) = 1 - H(x-1)$, $x \in [0,3]$, $\nu = 0$:

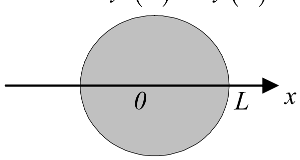
$$a_n = \frac{\int_0^3 x dx}{N_{0,0}^2} = \frac{1}{9}$$

$$a_n = \frac{\int_0^3 x J_0(\lambda_n x) dx}{N_{0,n}^2} = \frac{L^2}{2} \frac{J_1(\lambda_n)}{\lambda_n J_0^2(\lambda_n L)} = \frac{2}{9} \frac{J_1(\lambda_n)}{\lambda_n J_0^2(3\lambda_n)}$$

Maple solutions: $\nu = 0$ *SF-1-2-0.mws*
$\nu = 1$ *SF-1-2-1.mws*

Example 5 *(III Robin boundary condition)*

$$y'(L) + Hy(L) = 0$$



Consider an orthogonal set obtained as a solution of the Robin problem with Bessel functions of zero order $y_n^{(\nu)}(x) = J_\nu(\lambda_n x)$ where eigenvalues λ_n are positive roots of equation

$$-\lambda J_{\nu+1}(\lambda L) + \left(H + \frac{\nu}{L}\right) J_\nu(\lambda L) = 0$$

The squared norm of eigenfunctions can be calculated as

$$N_{\nu,n}^2 = \frac{L^2}{2}\left[\frac{H^2}{\lambda_n^2} + \left(1 - \frac{\nu^2}{\lambda_n^2 L^2}\right)\right] J_\nu^2(\lambda_n L)$$

Fourier-Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n y_n^{(\nu)}(x) \quad \text{where } a_n = \frac{\int_0^L x J_0(\lambda_n x) f(x) dx}{N_{\nu,n}^2}$$

Consider expansion of function $f(x) = 1 - H(x-1)$, $x \in [0,3]$, $H = 2$

Maple solution (for $\nu = 1$) *SF-1-3.mws*

SF-1.mws ***Example 3*** **Fourier-Bessel series** *I Dirichlet boundary condition* ν = order of Bessel Functions

```
> nu:=0;
```
$$\nu := 0$$

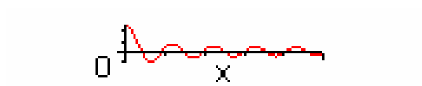
```
> L:=3;
```
$$L := 3$$

```
> f(x):=1-Heaviside(x-1);
```
$$f(x) := 1 - \mathrm{Heaviside}(x-1)$$

Characteristic equation:

```
> w(x):=BesselJ(nu,x*L);
```
$$w(x) := \mathrm{BesselJ}(0, 3x)$$

```
> plot(w(x),x=0..10);
```



Eigenvalues:

```
> lambda:=array(1..200);
```
$$\lambda := \mathrm{array}(1 .. 200, [\ ])$$

```
> n:=1: for m from 0 to 50 do
y:=fsolve(w(x)=0,x=m/2..(m+1)/2): if type(y,float) then
lambda[n]:=y: n:=n+1 fi od:
> for i to 2 do lambda[i] od;
```
$$0.8016085192$$
$$1.840026037$$

```
> N:=n-1;n:='n':i:='i':m:='m':y:='y':x:='x':
```
$$N := 24$$

Eigenfunctions:

```
> y[n]:=BesselJ(nu,lambda[n]*x);
```
$$y_n := \mathrm{BesselJ}(0, \lambda_n x)$$

Squared Norm:

```
>NY[n]:=int(x*y[n]^2,x=0..L):
NY[n]:=subs(BesselJ(nu,L*lambda[n])=0,NY[n]):
```

Fourier-Bessel coefficients:

```
> a[n]:=int(x*y[n]*f(x),x=0..L)/NY[n];
```
$$a_n := \frac{2}{9}\,\frac{\mathrm{BesselJ}(1, \lambda_n)}{\lambda_n\,\mathrm{BesselJ}(1, 3\lambda_n)^2}$$

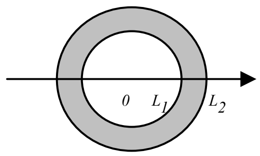
Fourier-Bessel series:

```
> u(x):=Sum(a[n]*y[n],n=1..N);
```
$$u(x) := \sum_{n=1}^{24} \left(\frac{2}{9}\,\frac{\mathrm{BesselJ}(1, \lambda_n)\,\mathrm{BesselJ}(0, \lambda_n x)}{\lambda_n\,\mathrm{BesselJ}(1, 3\lambda_n)^2} \right)$$

```
> u(x):=sum(a[n]*y[n],n=1..N):
> plot({f(x),u(x)},x=0..L,discont=true,color=black);
```

Orthogonal sets for annular domain

$L_1 \le x \le L_2$



Consider BE in the finite circle

$$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right) y = 0\,, \quad x \in \left(L_1, L_2\right)$$

with homogeneous boundary conditions:

$$\left[-k_1 \frac{dy}{dx} + h_1 y\right]_{x=L_1} = 0 \qquad H_1 = \frac{h_1}{k_1}$$

$$\left[k_2 \frac{dy}{dx} + h_2 y\right]_{x=L_2} = 0 \qquad H_2 = \frac{h_2}{k_2}$$

The general solution is given by

$$y(x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

A BVP for BE in the finite domain according to the Sturm-Liouville theorem generates an infinite set of eigenvalues λ_n and corresponding eigenfunctions $y_n(x)$ orthogonal with the weight function $p(x) = x$. A particular form of the orthogonal set depends on the type of boundary conditions.

Consider a case when both boundary conditions are of Dirichlet type:

Example 6 *(Dirichlet-Dirichlet boundary conditions)*

Boundary conditions:

$$y\big|_{x=L_1} = 0$$

$$y\big|_{x=L_2} = 0$$

Apply boundary conditions to the general solution of BE:

$$c_1 J_\nu(\lambda L_1) + c_2 Y_\nu(\lambda L_1) = 0$$

$$c_1 J_\nu(\lambda L_2) + c_2 Y_\nu(\lambda L_2) = 0$$

This is a homogeneous system of two linear algebraic equations for c_1 and c_2. Rewrite it in the matrix form

$$\begin{bmatrix} J_\nu(\lambda L_1) & Y_\nu(\lambda L_1) \\ J_\nu(\lambda L_2) & Y_\nu(\lambda L_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We are looking for non-trivial solution of BVP, i.e. both coefficients in general solution cannot be zero

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A homogeneous linear system has a non-trivial solution only if the determinant of the system matrix is equal to zero:

Equation for eigenvalues λ_n :

$$\boxed{det \begin{bmatrix} J_\nu(\lambda L_1) & Y_\nu(\lambda L_1) \\ J_\nu(\lambda L_2) & Y_\nu(\lambda L_2) \end{bmatrix} = J_\nu(\lambda L_1) Y_\nu(\lambda L_2) - J_\nu(\lambda L_2) Y_\nu(\lambda L_1) = 0}$$

The roots of this equation yield the eigenvalues λ_n for which BVP has non-trivial solutions $y_n(x)$ (eigenfunctions). Oscillatory property of Bessel functions provides an infinite set of eigenvalues λ_n and corresponding eigenfunctions are

$$y_n(x) = c_{1,n} J_\nu(\lambda_n x) + c_{2,n} Y_\nu(\lambda_n x)$$

Determine now the coefficients $c_{1,n}$ and $c_{2,n}$ from a system where eigenvalues are substituted

$$\begin{bmatrix} J_\nu(\lambda_n L_1) & Y_\nu(\lambda_n L_1) \\ J_\nu(\lambda_n L_2) & Y_\nu(\lambda_n L_2) \end{bmatrix} \begin{bmatrix} c_{1,n} \\ c_{2,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because a linear system has a singular matrix, solutions for $c_{1,n}$ and $c_{2,n}$ are linearly dependent and can be determined just from one equation, let it be the second one

$$c_{1,n} J_\nu(\lambda_n L_2) + c_{2,n} Y_\nu(\lambda_n L_2) = 0$$

one of the unknowns in this equation is a free parameter, choose

$c_{1,n} = \dfrac{1}{J_\nu(\lambda_n L_2)}$, then $c_{2,n} = \dfrac{-1}{Y_\nu(\lambda_n L_2)}$

Then eigenfunctions have the form:

Eigenfunctions

$$\boxed{y_n(x) = \frac{J_\nu(\lambda_n x)}{J_\nu(\lambda_n L_2)} - \frac{Y_\nu(\lambda_n x)}{Y_\nu(\lambda_n L_2)}}$$

The norm of eigenfunctions is given by:

$$\begin{aligned} N_{\nu,n}^2 &= \int_{L_1}^{L_2} x y_n^2(x) dx \\ &= \int_{L_1}^{L_2} x \left[\frac{J_\nu(\lambda_n x)}{J_\nu(\lambda_n L_2)} - \frac{Y_\nu(\lambda_n x)}{Y_\nu(\lambda_n L_2)} \right]^2 dx \\ &= \frac{1}{J_\nu^2(\lambda_n L_2)} \int_{L_1}^{L_2} x J_\nu^2(\lambda_n x) dx + \frac{1}{Y_\nu^2(\lambda_n L_2)} \int_{L_1}^{L_2} x Y_\nu^2(\lambda_n x) dx - \\ &\qquad - \frac{1}{J_\nu(\lambda_n L_2) Y_\nu(\lambda_n L_2)} \int_{L_1}^{L_2} x J_\nu(\lambda_n x) Y_\nu(\lambda_n x) dx \\ &= \ldots \quad \text{express in terms of } J_{\nu+1}, \ldots \end{aligned}$$

Summary: For an annular domain with boundary conditions:

$$y\big|_{x=L_1} = 0$$

$$y\big|_{x=L_2} = 0$$

Eigenvalues λ_n are positive roots of the characteristic equation

$$J_\nu(\lambda L_1) Y_\nu(\lambda L_2) - J_\nu(\lambda L_2) Y_\nu(\lambda L_1) = 0$$

The eigenfunctions are

$$y_n(x) = \frac{J_\nu(\lambda_n x)}{J_\nu(\lambda_n L_2)} - \frac{Y_\nu(\lambda_n x)}{Y_\nu(\lambda_n L_2)}$$

Fourier-Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

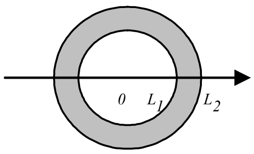
where $$a_n = \frac{\int_0^L x y_n(x) f(x) dx}{\int_0^L x y_n^2(x) dx} = \frac{\int_0^L x J_\nu(\lambda_n x) f(x) dx}{N_{\nu,n}^2}$$

<u>Maple examples</u>: $\nu = 0$ *SF-AD-1-0.mws*
$\nu = 1$ *SF-AD-1-1.mws*

$L_1 = 2$, $L_2 = 5$ $f(x) = 1 - H(x-3)$

Consider a case when both boundary conditions are of Robin type:

Example 7 (*Robin-Robin boundary conditions*) *SF-5.mws*



Consider BE in the annular domain

$$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right)y = 0\,, \quad x \in (L_1, L_2)$$

with homogeneous boundary conditions:

$$\left[-k_1 \frac{dy}{dx} + h_1 y\right]_{x=L_1} = 0 \qquad H_1 = \frac{h_1}{k_1}$$

$$\left[k_2 \frac{dy}{dx} + h_2 y\right]_{x=L_2} = 0 \qquad H_2 = \frac{h_2}{k_2}$$

The general solution is given by

$$y(x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

The derivative of the general solution (use chain rule and differential identities)

$$\frac{d}{dx} y(x) = c_1 \lambda\left[-J_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} J_\nu(\lambda x)\right] + c_2 \lambda\left[-Y_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} Y_\nu(\lambda x)\right]$$

Substitute into boundary conditions:

$x = L_1$
$$-c_1\lambda\left[-J_{\nu+1}(\lambda L_1) + \frac{\nu}{\lambda L_1} J_\nu(\lambda L_1)\right] - c_2\lambda\left[-Y_{\nu+1}(\lambda L_1) + \frac{\nu}{\lambda L_1} Y_\nu(\lambda L_1)\right] + c_1 H_1 J_\nu(\lambda L_1) + c_2 H_1 Y_\nu(\lambda L_1) = 0$$

$x = L_2$
$$c_1\lambda\left[-J_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} J_\nu(\lambda L_2)\right] + c_2\lambda\left[-Y_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} Y_\nu(\lambda L_2)\right] + c_1 H_2 J_\nu(\lambda L_2) + c_2 H_2 Y_\nu(\lambda L_2) = 0$$

Collect terms

$x = L_1$
$$c_1\left[\lambda J_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1}\right) J_\nu(\lambda L_1)\right] + c_2\left[\lambda Y_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1}\right) Y_\nu(\lambda L_1)\right] = 0$$

$x = L_2$
$$c_1\left[-\lambda J_{\nu+1}(\lambda L_2) + \left(H_2 + \frac{\nu}{L_2}\right) J_\nu(\lambda L_2)\right] + c_2\left[-\lambda Y_{\nu+1}(\lambda L_2) + \left(H_2 - \frac{\nu}{L_2}\right) Y_\nu(\lambda L_2)\right] = 0$$

Denote:

$$a_{11} = \left[\lambda J_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1}\right) J_\nu(\lambda L_1)\right]$$

$$a_{12} = \left[\lambda Y_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1}\right) Y_\nu(\lambda L_1)\right]$$

$$a_{21} = \left[-\lambda J_{\nu+1}(\lambda L_2) + \left(H_2 + \frac{\nu}{L_2}\right) J_\nu(\lambda L_2)\right]$$

$$a_{22} = \left[-\lambda Y_{\nu+1}(\lambda L_2) + \left(H_2 - \frac{\nu}{L_2}\right) Y_\nu(\lambda L_2)\right]$$

Then a system for coefficients has the following matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A necessary condition for a system to have a non-trivial solution is

$$det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = 0$$

it yields a characteristic equation for values of the parameter λ for which the BVP has a non-trivial solution:

Equation for eigenvalues λ_n:

$$\left[\lambda J_{\nu+1}(\lambda L_1)+\left(H_1-\frac{\nu}{L_1}\right)J_\nu(\lambda L_1)\right]\left[-\lambda Y_{\nu+1}(\lambda L_2)+\left(H_2+\frac{\nu}{L_2}\right)Y_\nu(\lambda L_2)\right]$$
$$-\left[\lambda Y_{\nu+1}(\lambda L_1)+\left(H_1-\frac{\nu}{L_1}\right)Y_\nu(\lambda L_1)\right]\left[-\lambda J_{\nu+1}(\lambda L_2)+\left(H_2-\frac{\nu}{L_2}\right)J_\nu(\lambda L_2)\right]=0$$

The positive roots of this equation provide an infinite set of eigenvalues λ_n. Then for the determined eigenvalues λ_n, coefficients $c_{1,n}$ and $c_{2,n}$ can be found from one of the equations of the system (choose the second one):

$a_{21}c_1 + a_{22}c_2 = 0$

One of the coefficients can be taken as a free parameter, choose

$c_1 = \dfrac{1}{a_{21}}$, then $c_2 = \dfrac{1}{a_{22}}$

With determined coefficients, solutions of the BVP $y_n(x)$ (eigenfunctions) have the form:

Eigenfunctions:

$$y_n(x) = \frac{J_\nu(\lambda_n x)}{a_{21,n}} - \frac{Y_\nu(\lambda_n x)}{a_{22,n}}$$

$$= \frac{J_\nu(\lambda_n x)}{\left[-\lambda_n J_{\nu+1}(\lambda_n L_2)+\left(H_2+\frac{\nu}{L_2}\right)J_\nu(\lambda_n L_2)\right]} - \frac{Y_\nu(\lambda_n x)}{\left[-\lambda_n Y_{\nu+1}(\lambda_n L_2)+\left(H_2-\frac{\nu}{L_2}\right)Y_\nu(\lambda_n L_2)\right]}$$

The norm of the eigenfunctions is determined by the integral

$$N_{\nu,n}^2 = \int_{L_1}^{L_2} x y_n^2(x)\,dx$$

Fourier-Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

where $a_n = \dfrac{\int_{L_1}^{L_2} x y_n(x) f(x)\,dx}{\int_{L_1}^{L_2} x y_n^2(x)\,dx} = \dfrac{\int_{L_1}^{L_2} x y_n^2(x) f(x)\,dx}{N_{\nu,n}^2}$

<u>Maple example</u> $\nu = 0$ *SF-AD-9-0.mws*
$\nu = 1$ *SF-AD-9-1.mws*

$L_1 = 2$, $L_2 = 5$
$H_1 = 2$, $H_2 = 3$
$f(x) = 1 - H(x-3)$

5.7 Legendre Functions

1. Legendre Equation

Separation of variables of the Laplacian in a spherical coordinate system yields a group of ODE one of which has the form

$$\left[\left(1-x^2\right)y'\right]' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$$

where m and n are separation constants.

This equation is called **Legendre's associated differential equation**.

Solution of this equation includes Legendre's associated functions of degree n and of order m of the 1st and the 2nd kind $P_n^m(x)$ and $Q_n^m(x)$.

When $m=0$ (in a case when the Laplacian does not depend on the variable ϕ), equation is called the **Legendre's differential equation**

$$\left[\left(1-x^2\right)y'\right]' + n(n+1)y = 0$$

Solution of this equation include Legendre's functions of degree n of the 1st and the 2nd kind $P_n(x)$ and $Q_n(x)$.

2. Solution of Legendre Equation

Consider the Legendre differential equation rewritten in standard form

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0 \qquad x \in R$$

This equation has two singular points $x = \pm 1$, all other points are ordinary points. We will apply a power-series solution method around the ordinary point $x=0$ (the interval of convergence for this solution is $(-1,1)$). Assume that the solution is represented by a power series

$$y = \sum_{k=0}^{\infty} a_k x^k$$

then derivatives of the solution are

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Substitute them into equation

$$\left(1-x^2\right)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=1}^{\infty} k a_k x^{k-1} + n(n+1)\sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=1}^{\infty} 2k a_k x^k + n(n+1)\sum_{k=0}^{\infty} a_k x^k = 0$$

(change $k=k-2$ in the first term)

$$\sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2} x^k - \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=1}^{\infty} 2k a_k x^k + n(n+1)\sum_{k=0}^{\infty} a_k x^k = 0$$

$$2\cdot 1\cdot a_2 + 2\cdot 3\cdot a_3 x - 2\cdot a_1 x + n(n+1)a_0 + n(n+1)a_1 x + \sum_{k=2}^{\infty}\left\{a_k\left[-k(k-1) - 2k + n(n+1)\right] + (k+1)(k+2)a_{k+2}\right\} = 0$$

$$n(n+1)a_0 + 1\cdot 2\cdot a_2 + \left\{2\cdot 3\cdot a_3 + \left[n(n+1) - 2\right]a_1\right\}x + \sum_{k=2}^{\infty}\left\{a_k\left[n(n+1) - k(k+1)\right] + (k+1)(k+2)a_{k+2}\right\} = 0$$

Using the comparison theorem, determine the relation for coefficients:

$$a_2 = \frac{-n(n+1)}{1\cdot 2}a_0$$

$$a_3 = \frac{2-n(n+1)}{2\cdot 3}a_1 = \frac{(n-1)(n+2)}{2\cdot 3}a_1$$

$$a_{k+2} = \frac{k(k+1)-n(n+1)}{(k+1)\cdot(k+2)}a_k = \frac{-(n-k)(n+k+1)}{(k+1)\cdot(k+2)}a_k \qquad k = 2,3,...$$

Coefficients a_0 and a_1 are arbitrary, consider them to be the parameters for the general solution and collect the terms corresponding to these coefficients, then the power series solution of the Legendre Equation becomes

$$\begin{aligned} y(x) &= a_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{n(n-2)(n+1)(n+3)}{4!}x^4-\frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6+...\right] \\ &+a_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5-\frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7+...\right] \\ &= a_0L_{n,1}(x)+a_1L_{n,2}(x) \end{aligned}$$

Choose a sequence of non-negative values of $n = 0,1,2,...$ then corresponding solutions are (note, that in the solution all terms except for finite number alternatingly disappear: if $n=2k$ is even then in the first series all terms with multiple $(n-2k)$ disappear, if $n=2k+1$ is odd then in the second series all terms with multiple $(n-2k-1)$ disappear, and they become the finite polynomials)

	$L_{1,n}(x)$	$L_{2,n}(x)$
$n=0$	a_0	$a_1L_{2,0}(x)$
$n=1$	$a_0L_{1,1}(x)$	a_1x
$n=2$	$a_0(1-3x^2)$	$a_1L_{2,2}(x)$
$n=3$	$a_0L_{1,3}(x)$	$a_1\left(x-\frac{5}{3}x^3\right)$
…		
Choose	$a_0 = (-1)^{\frac{n}{2}}\frac{n!}{2^n\left(\frac{n}{2}!\right)^2}$	$a_1 = (-1)^{\frac{n-1}{2}}\frac{(n+1)!}{2^n\left(\frac{n-1}{2}!\right)^2\left(\frac{n+1}{2}!\right)^2}$

Then Legendre functions of the 1st kind for different values of parameter n generate the following set of polynomials

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$P_3(x) = \frac{5}{3}x^3 - \frac{3}{2}x$

$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

…

which are called **Legendre polynomials**. Because Legendre polynomials are solutions of the separated Laplace equation in spherical coordinates, they are also called **spherical harmonics** (and the method of solution in terms of Legendre functions is called correspondingly the Method of Spherical Harmonics). Recall that this system of polynomials up to scalar multiple was also obtained from orthogonalization of the linear independent set of monoms $\{1,x,x^2,x^3,...\}$ on the interval $[-1,1]$.

Recurrence formula

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$$

Rodrigues' formula

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^2}\left(x^2-1\right)^n$$

Orthogonality of Legendre polynomials

Legendre polynomials are orthogonal in the interval $[-1,1]$ with the weight function $p(x)=1$

$$\int_{-1}^{1}P_m(x)P_n(x)dx=\begin{cases}0 & m\neq n\\ \dfrac{2}{2n+1} & m=n\end{cases}$$

<u>Fourier-Legendre series</u>

Legendre polynomials can be used for expansion of the function $f(x)$, $x\in[-1,1]$ in the Fourier-Legendre series:

$$\boxed{f(x)=\sum_{n=0}^{\infty}c_nP_n(x)}$$

where expansion coefficients are

$$\boxed{c_n=\frac{\int_{-1}^{1}f(x)P_n(x)dx}{\int_{-1}^{1}P_n^2(x)dx}=\left(n+\frac{1}{2}\right)\int_{-1}^{1}f(x)P_n(x)dx}$$

Example 8 *(expansion in Fourier-Legendre series (spherical harmonics))*

$f(x)=H(x)\quad x\in[-1,1]$

Maple Solution: *SF-8.mws*

$$f(x)=\sum_{n=0}^{\infty}\left[\left(n+\frac{1}{2}\right)\int_{0}^{1}P_n(x)dx\right]P_n(x)$$

Integral transform

The integral transform based on the Fourier-Legendre expansion

$$\bar{f}_n=\int_{-1}^{1}f(x)K_n(x)dx$$

with inverse transform

$$f(x)=\sum_{n=0}^{\infty}\bar{f}_nK_n(x)$$

where the kernel of the integral transform is defined as a normalized Legendre function

$$K_n(x)=\sqrt{n+\frac{1}{2}}P_n(x)$$

SF-8.mws **Example 8 Fourier-Legendre Series**

```
> restart;
> with(orthopoly);
```

$$[G, H, L, P, T, U]$$

```
> for n from 0 to 6 do P(n,x) od;
```

$$1$$

$$x$$

$$-\frac{1}{2}+\frac{3x^2}{2}$$

$$\frac{5}{2}x^3-\frac{3}{2}x$$

$$\frac{3}{8}+\frac{35}{8}x^4-\frac{15}{4}x^2$$

$$\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x$$

$$-\frac{5}{16}+\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2$$

```
> f(x):=Heaviside(x);
```

$$f(x) := \text{Heaviside}(x)$$

Fourier-Legendre coefficients:

```
> c[n]:=(n+1/2)*int(f(x)*P(n,x),x=-1..1);
```

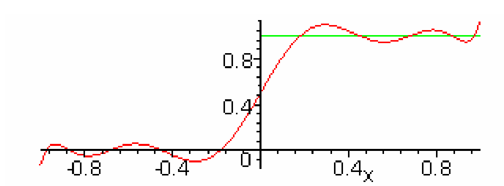
$$c_n := \left(n+\frac{1}{2}\right)\int_0^1 P(n,x)\,dx$$

Fourier-Legendre series:

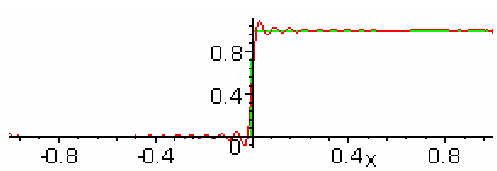
```
> u(x):=sum(c[n]*P(n,x),n=0..10);
```

$$u(x) := \frac{1}{2}+\frac{218295}{65536}x-\frac{315315}{16384}x^3+\frac{1702701}{32768}x^5-\frac{984555}{16384}x^7+\frac{1616615}{65536}x^9$$

```
> plot({f(x),u(x)},x=-1..1);
```




```
> u(x):=sum(c[n]*P(n,x),n=0..100):
> plot({f(x),u(x)},x=-1..1);
```



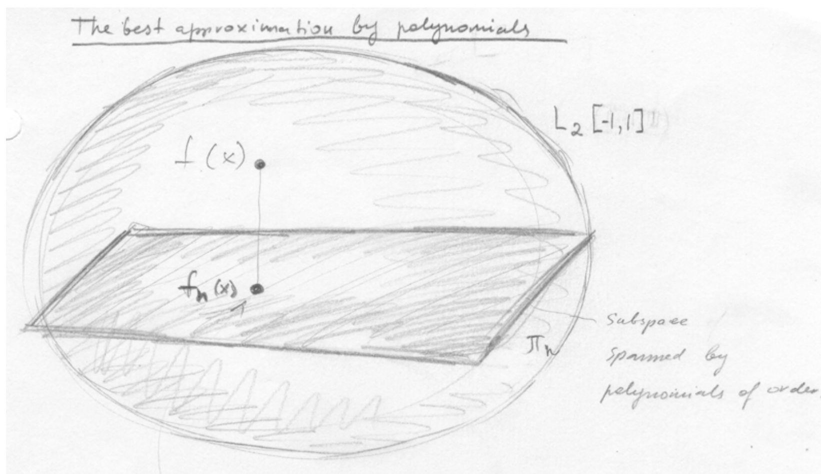
The Best Approximation by Polynomials

Consider a vector space of square integrable functions $L_2[-1,1]$. The span of all polynomials of order n is a subspace of $L_2[-1,1]$. Call it π_n.

Let $f(x) \in \pi_n$ and let

$$f_n(x) = \sum_{k=1}^{n} c_n P_n(x)$$

be the n^{th} partial sum of the Fourier-Legendre expansion of the function $f(x)$ in $[-1,1]$.



Then $f_n(x)$ provides the best approximation of the function $f(x)$ by the n^{th} order polynomials, i.e. function $f_n(x)$ is the closest to the function $f(x)$ among the functions in π_n in the sense that it minimizes the distance

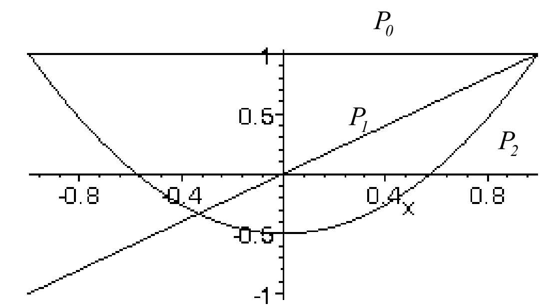
$$\left\| f(x) - p_n(x) \right\| = (f - p_n, f - p_n) = \int_{-1}^{1} (f - p_n)^2 \, dx$$

Graphs of Legendre polynomials

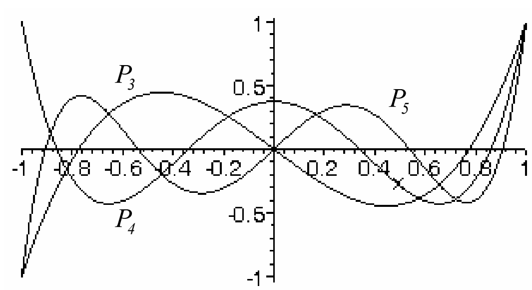
Legendre-1.mws   **Legendre polynomials**

```
> restart;
> with(orthopoly):
```

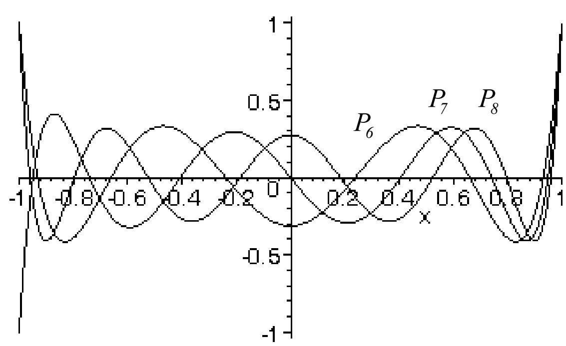
```
> plot({P(0,x),P(1,x),P(2,x)},x=-1..1);
```




```
> plot({P(3,x),P(4,x),P(5,x)},x=-1..1);
```




```
> plot({P(6,x),P(7,x),P(8,x)},x=-1..1);
```



5.8 Exercises:

1) Show

$$\sin x\,\delta(x-n\pi)=0$$
$$\sin x\,\delta'(x-n\pi)=(-1)^{n+1}\delta(x-n\pi)$$

(by multiplying both sides by an arbitrary differentiable function and integrating)

2) Show

$$\int_{-\infty}^{\infty}\delta(3x+6)f(x)dx=\frac{f(-2)}{3}$$

In general

$$\int_{-\infty}^{\infty}\delta[g(x)]f(x)dx=\frac{f(x_0)}{|g'(x_0)|}\quad \text{where } g(x_0)=0$$

3) Solve the IVP and sketch the solution curves (use Maple and Laplace transform):

$$y''-y=(t-2)H(t-2)\qquad y(0)=0\qquad y'(0)=0$$

$$y''+2y'+y=3\delta(t-2)\qquad y(0)=1\qquad y'(0)=1$$

4) **Sign function** is defined as

$$sgn(x)=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$$

a) Express $sgn(x)$ in terms of Heaviside step function $H(x)$
b) Express Heaviside step function $H(x)$ in terms of $sgn(x)$
c) Calculate $\frac{d}{dx}sgn(x)$
d) Sketch the graph of $sgn(x-3)$

5) Investigate convergence of the Bessel function $J_\nu(x)$ using the ratio test (see supplemental materials)

6) Use term-by-term differentiation (why can we do it?) to show

$$\frac{d}{dx}J_0(x)=-J_1(x)$$

7) Solve in terms of Bessel functions

$$x^2 y'' + (x-2)y = 0$$

8) Finish Examples 4 and 5 in section 6.12 Orthogonality of Bessel Functions

9) Hermite's differential equation with parameter λ is

$$\boxed{y'' - 2xy' + \lambda y = 0} \qquad x \in (-\infty, \infty),\ \lambda \in \mathbb{R} \qquad \text{(HE)}$$

a) Solve the HE by the power series method

b) Consider two linearly independent solutions

$y_1(x) = \ldots$

$y_2(x) = \ldots$

which include parameter λ

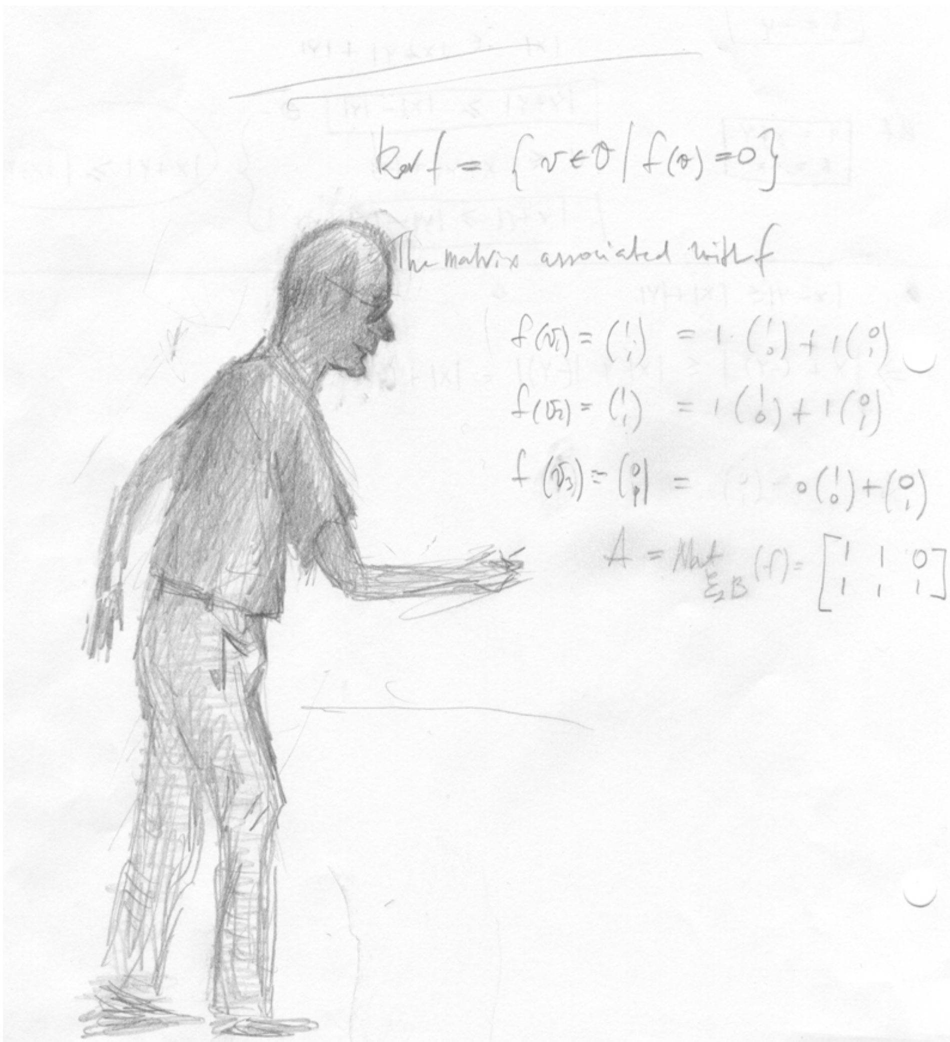
c) If λ is a non-negative even integer, $\lambda = 0, 2, 4, \ldots, 2n, \ldots$, then the series terminates, and one obtains, alternating for y_1 and y_2, polynomials of degree n, which are called Hermitian polynomials $H_n(x)$. Write them in traditional form in which the coefficient of x^n is equal to 2^n (the second solution is not polynomial).

d) Rewrite HE in self-adjoint form and determine the weight function $w(x)$

e) Check if the HP are orthogonal with the weight function $w(x)$ over $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) w(x)\, dx = 0 \quad \text{if} \quad m \neq n$$

f) Give an example of function representation into Fourier-Hermite series


The matrix associated with f