Rotating Disc Fluid Mechanics

ChEn 533

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Flow of a Viscous Liquid on a Rotating Disk*

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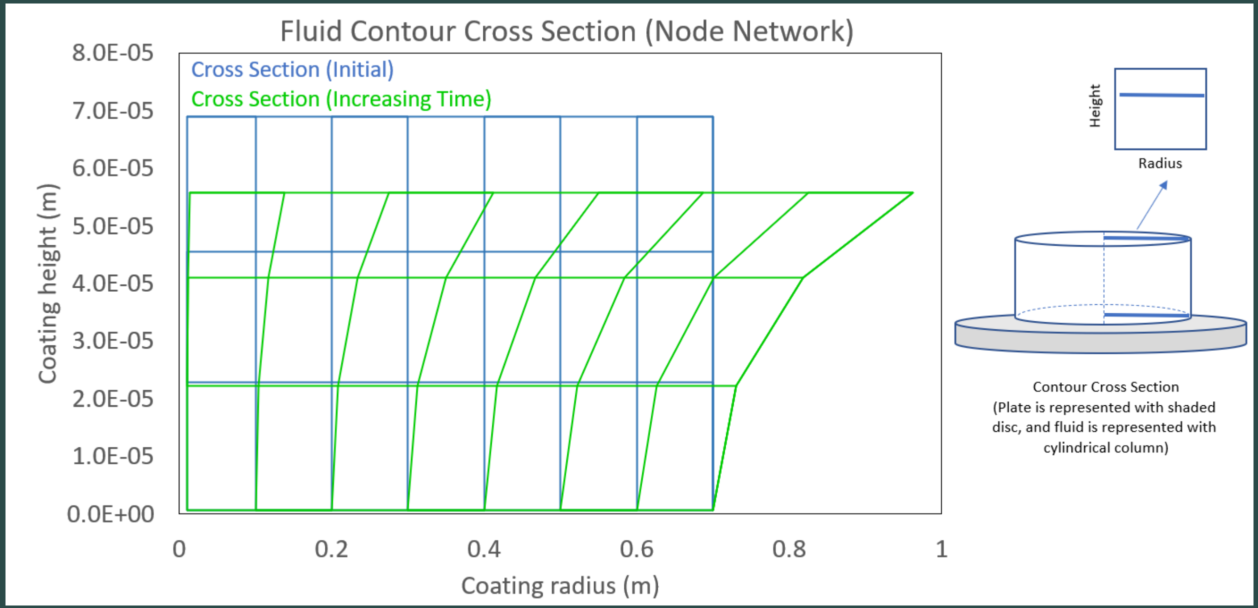
Introduction



Foundational Assumptions	
1	Semi-infinite flat plate (the plate radius is larger than the outermost extent of fluid)
2	Gravitational forces are negligible compared to centrifugal forces (the plate spins fast)
3	Radially symmetric initial (and transient) fluid distribution
4	Viscosity is independent of shear rate (Newtonian Fluid)
5	Shear resistance is only significant in horizontal planes (relatively thin layers)
6	Radial velocity is sufficiently low to neglect Coriolis forces.
7	The top of the fluid film is exposed to air which exerts negligible shear forces.

Methods

$$-\mu\left(\frac{\partial^2 v_r}{\partial z^2}\right) = \rho(\omega^2 r)$$

Methods: backtracking

Equation of motion, cylindrical coordinates

$$\rho\left(\frac{Dv}{Dt}\right) = -\nabla P - (\nabla \cdot \tau) + \rho g$$

$$F_{gv} = \rho g$$

Gravitational body force

$$F = ma$$

$$F_g = mg$$

$$F_{cv} = \rho(\omega^2 r)$$

Centrifugal body force

$$F_c = m\left(\frac{v^2}{r}\right)$$

$$F_c = m(\omega^2 r)$$

Methods: backtracking

$$\rho\left(\frac{Dv}{Dt}\right) = -\nabla P - (\nabla \cdot \tau) + \rho(\omega^2 r)$$

Body forces: use F_{cv} instead of F_{gv}

$$\rho\left(\frac{Dv}{Dt}\right) = -\nabla P + \mu\nabla^2 v + \rho(\omega^2 r)$$

Newtonian Fluid with constant density

$$-\mu\left(\frac{\partial^2 v_r}{\partial z^2}\right) = \rho(\omega^2 r)$$

Reference frame of moving plate as v = 0

Negligible Pressure gradient

$$-\mu\left(\frac{\partial^2 v_r}{\partial z^2}\right) = \rho(\omega^2 r)$$

$$-\mu\left(\frac{\partial^2 v_r}{\partial z^2}\right) = \rho(\omega^2 r)$$ Integrate

$$\frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial z}\right) = -\frac{\rho(\omega^2 r)}{\mu}$$

$$\int \partial\left(\frac{\partial v_r}{\partial z}\right) = -\int \frac{\rho(\omega^2 r)}{\mu}\partial z$$

Boundary Conditions

$$\frac{\partial v_r}{\partial z} = 0 \quad @ \quad z = h$$

$$\frac{\partial v_r}{\partial z} = -\frac{\rho(\omega^2 r)}{\mu}z + C_1$$

$$0 = -\frac{\rho(\omega^2 r)}{\mu}h + C_1$$

$$\int \partial v_r = -\int\left(\frac{\rho(\omega^2 r)}{\mu}z + C_1\right)\partial z$$

$$C_1 = \frac{\rho(\omega^2 r)}{\mu}h$$

$$v_r = -\frac{\rho(\omega^2 r)}{2\mu}z^2 + C_1 z + C_2$$

$$v_r = 0 \quad @ \quad z = 0$$

$$0 = 0 + 0 + C_2$$

$$C_2 = 0$$

$$v_r = -\frac{\rho(\omega^2 r)}{2\mu}z^2 + \frac{\rho(\omega^2 r)}{\mu}hz$$ Sub in C_1 and C_2

$$v_r = \frac{\rho(\omega^2 r)}{\mu}\left(-\frac{z^2}{2} + hz\right)$$ Rewrite to simplify

$$v_r = \frac{\rho(\omega^2 r)}{\mu}\left(-\frac{z^2}{2} + hz\right)$$

$$q = \int_0^h v_r dz$$

$$q = \frac{\rho\omega^2 r h^3}{3\mu}$$

$$r\frac{\partial h}{\partial t} = \frac{\partial(rq)}{\partial \mathrm{r}}$$

$$\frac{\partial h}{\partial t} = -\frac{\rho(\omega^2)}{3\mu r}\frac{\partial}{\partial r}(r^2 h^3)$$

$$\frac{\partial}{\partial r}(r^2 h^3) = h^3\frac{\partial}{\partial r}(r^2) + r^2\frac{\partial}{\partial r}(h^3)$$

$$\frac{\partial h}{\partial t} = \frac{2\rho(\omega^2)h^3}{3\mu} - \frac{3\rho(\omega^2)rh^2}{3\mu}\frac{\partial h}{\partial r}$$ PDE

$$-\frac{2\rho(\omega^2)h^3}{3\mu} = \frac{\partial h}{\partial t} + \frac{\rho(\omega^2)rh^2}{\mu}\frac{\partial h}{\partial r}$$

Rearrange

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial r}\frac{dr}{dt}$$

Use total derivative definition

$$\frac{dh}{dt} = -\frac{2\rho(\omega^2)h^3}{3\mu}$$

$$\frac{dr}{dt} = \frac{\rho(\omega^2)rh^2}{\mu}$$

2 ODEs from 1 PDE

$$\int \frac{1}{h^3} dh = -\frac{2\rho(\omega^2)}{3\mu} dt$$

Boundary Condition

$$\frac{-0.5}{h^2} = -\frac{2\rho(\omega^2)}{3\mu} t + C_1$$

$$h = h_o \ @ \ t = 0$$

$$\frac{-0.5}{h_o^2} = C_1$$

Sub in C_1

$$\frac{-0.5}{h^2} = -\frac{2\rho(\omega^2)}{3\mu} t - \frac{0.5}{h_o^2}$$

$$\frac{-0.5}{1} = h^2 \left(-\frac{2\rho(\omega^2)}{3\mu} t - \frac{0.5}{h_o^2} \right)$$

$$\frac{-0.5}{\left(-\frac{2\rho(\omega^2)}{3\mu} t - \frac{0.5}{h_o^2} \right)} = h^2$$

$$h^2 = \frac{1}{\left(\frac{4\rho(\omega^2)}{3\mu} t + \frac{1}{h_o^2} \right)}$$

$$\frac{dr}{dt} = \frac{\rho(\omega^2) r h_o^2}{\mu\left(\frac{4\rho(\omega^2)h_o^2}{3\mu}t + 1\right)}$$

$$\int \frac{1}{r} dr = \int \frac{\rho(\omega^2) h_o^2}{\mu\left(\frac{4\rho(\omega^2)h_o^2}{3\mu}t + 1\right)} dt$$

$$A = \frac{\rho(\omega^2)h_o^2}{\mu} \qquad B = \frac{4\rho(\omega^2)h_o^2}{3\mu}$$

$$\int \frac{1}{r} dr = A \int \frac{1}{(Bt+1)} dt$$

$$\ln(r) = \frac{A}{B}\ln(Bt+1) + C_1$$

Boundary Condition

$r = r_o \;\; @ \;\; t = 0$

$$\ln(r_o) = C_1$$

$$C_1 = \ln(r_o)$$

$$\ln(r) = \frac{A}{B}\ln(Bt+1) + \ln(r_o)$$

Sub in C_1

$$\frac{\ln(r)}{\ln(r_o)} = \frac{A}{B}\ln(Bt+1)$$

Rearrange

$$\frac{r}{r_o} = (Bt+1)^{\frac{A}{B}}$$

$$r = r_o\left(\frac{4\rho(\omega^2)h_o^2}{3\mu}t + 1\right)^{3/4}$$

$$\frac{\partial h}{\partial t} = \frac{2\rho(\omega^2)h^3}{3\mu} - \frac{3\rho(\omega^2)rh^2}{3\mu}\frac{\partial h}{\partial r}$$

PDE

$$\boxed{-\frac{2\rho(\omega^2)h^3}{3\mu}} = \frac{\partial h}{\partial t} + \boxed{\frac{\rho(\omega^2)rh^2}{\mu}\frac{\partial h}{\partial r}}$$

Rearrange

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial r}\frac{dr}{dt}$$

Use total derivative definition

$$\frac{dh}{dt} = -\frac{2\rho(\omega^2)h^3}{3\mu} \qquad \frac{dr}{dt} = \frac{\rho(\omega^2)rh^2}{\mu}$$

2 ODEs from 1 PDE

$$h = \frac{h_o}{\left(\frac{4\rho(\omega^2)h_o^2}{3\mu}t + 1\right)^{1/2}} \qquad r = r_o\left(\frac{4\rho(\omega^2)h_o^2}{3\mu}t + 1\right)^{3/4}$$

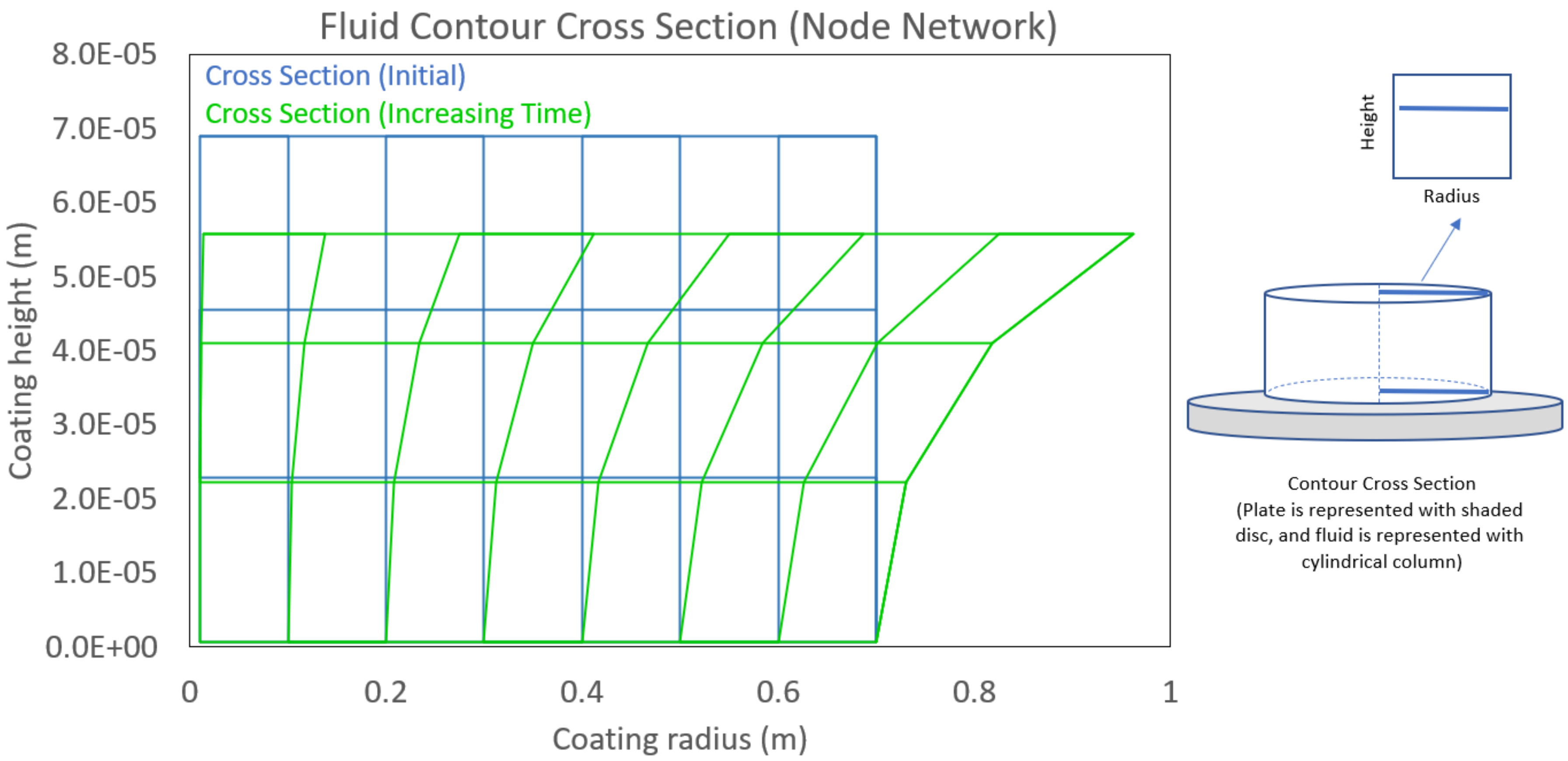
ODE Solutions for

$h = h_o \ @ \ t = 0$

$r = r_o \ @ \ t = 0$

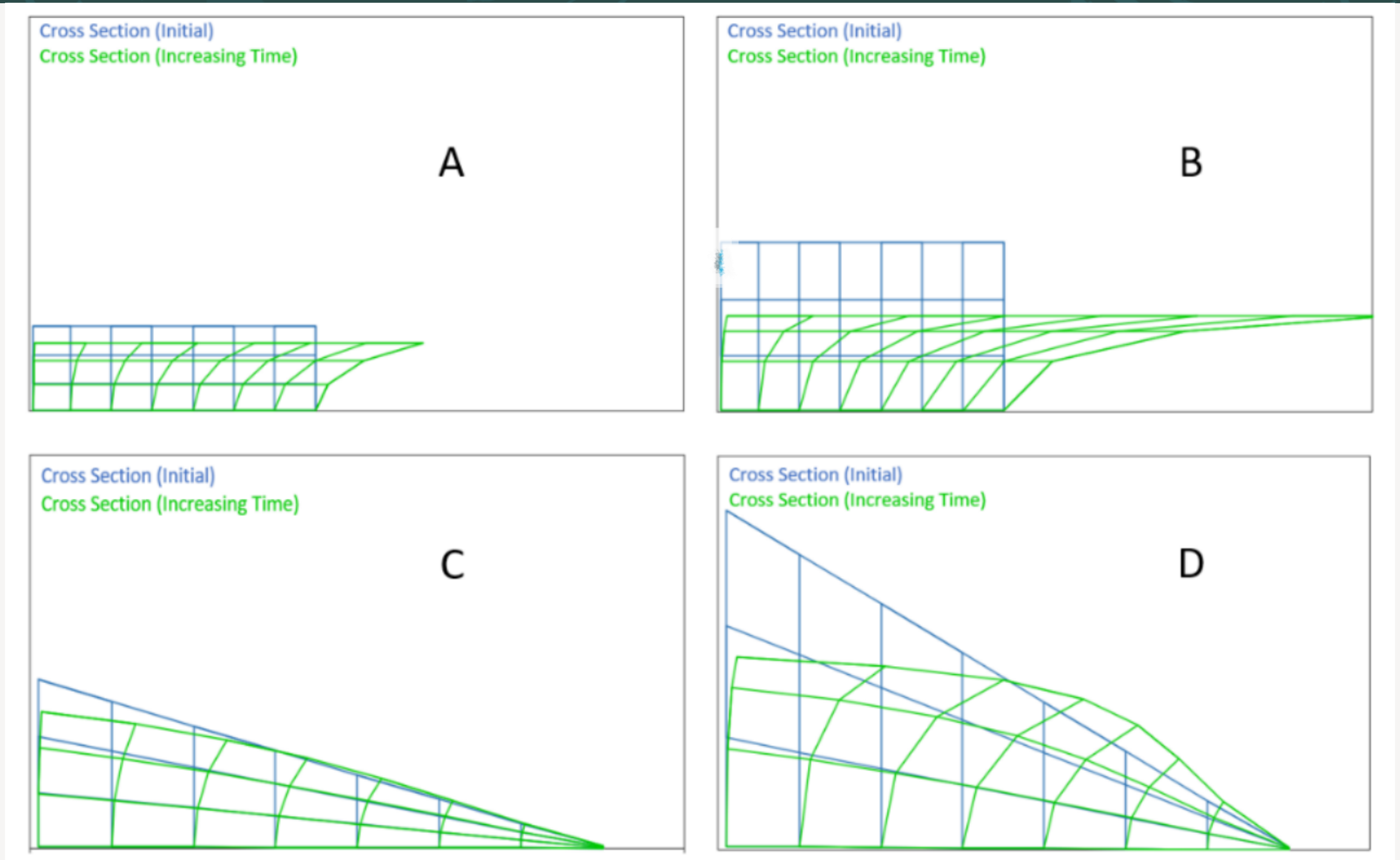
$$h = \frac{h_o}{(Kh_o^2t + 1)^{1/2}} \qquad r = r_o(Kh_o^2t + 1)^{3/4} \qquad K = \frac{4\rho(\omega^2)}{3\mu}$$

Results and Discussion

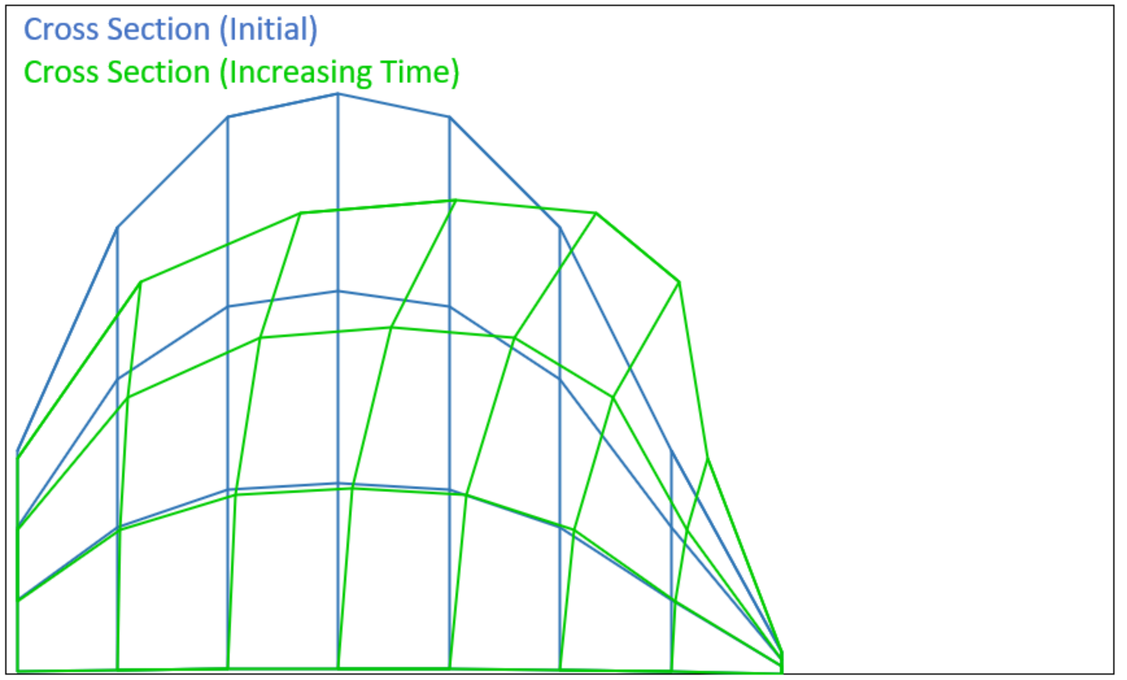


Contour Cross Section (Plate is represented with shaded disc, and fluid is represented with cylindrical column)

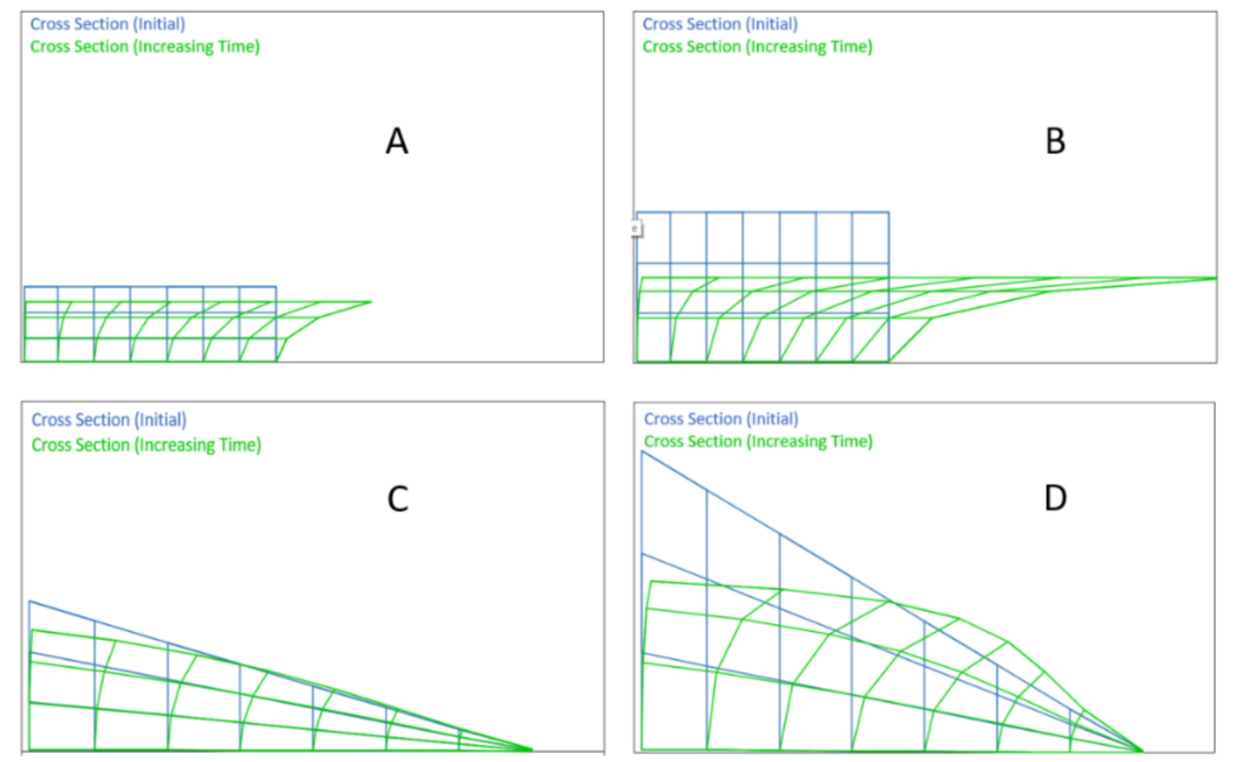
Results and Discussion (continued)



Conclusion



Cross Section (Initial)
Cross Section (Increasing Time)



Cross Section (Initial)
Cross Section (Increasing Time)
A
Cross Section (Initial)
Cross Section (Increasing Time)
B
Cross Section (Initial)
Cross Section (Increasing Time)
C
Cross Section (Initial)
Cross Section (Increasing Time)
D