Determining the Velocity Profile of Blood

## Introduction

The purpose of this paper is to determine the velocity profile of blood. This is made difficult as blood, despite its liquid appearance, is actually a multi-phase mixture of solids (red blood cells, white blood cells, and platelets) suspended in a liquid plasma which contains aqueous proteins, organic molecules, and minerals [6]. These components interact with one another in many different ways, making blood a viscoelastic real plastic. This behavior primarily comes from red blood cells which are characteristically elastic and can form three-dimensional structures called rouleaux at low shear rates [6]. Therefore, the non-Newtonian effects observed in blood are mostly evident at lower shear rates and shear thinning is experienced as the rouleaux begin to break apart under increasing shear rate. This unique and chimeric behavior can make modeling the velocity profile of blood difficult and has been the study of many different scientists for centuries.

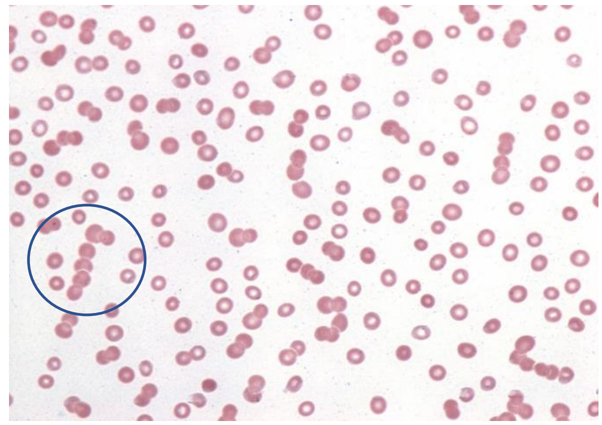

*Figure 1: Rouleaux clumps of red blood cells shown. This phenomenon is due to the discoid shape of the cells.*

Scientists have long studied the rheology of blood, including its velocity profile, as an understanding of blood flow can have many implications including improved treatment and diagnosis for patients with anemia, blood clots, etc.. One result of these studies was the well-known Poiseuille flow model, created by the famous French physician and physicist Jean Poiseuille in an effort to understand blood flow [3]. Although Poiseuille's model can represent blood flow in large arteries and veins where larger shear rates break up the rouleaux clumps making blood more Newtonian, models that represent the non-Newtonian nature of blood are required in order to represent smaller veins and capillaries. These include one created by Casson and two created by Henry Eyring. For a more complete list of different theories, please see source [6]. This paper will follow Merrill's approach [1], using the Casson derivations for the development for the velocity profile of blood.

## Methods

In order to determine the velocity profile of blood, an understanding and implementation of a Casson fluid's shear stress is required. Furthermore, one must create a force balance on a control volume in the fluid to obtain an expression for the shear stress as a function of radius, $r$. The expression for a Casson fluid and the force balance can be combined to create a differential equation which when integrated, considering the effects of yield stress, yields the velocity profile [3].

To understand the idea of the Casson representation of a non-Newtonian fluid, it is helpful to see *Figure 1* from Merrill's 1969 paper [1]. Here one can see how the square root of the shear rate, $\dot{\gamma}$, is linearly proportional to the square root of the shear stress, $\tau = \tau_{rz}$, in the non-Newtonian regime. Specifically, we can write:

$$\tau_{rz}^{1/2} = \tau_y^{1/2} + s\dot{\gamma}^{1/2} \quad (1)$$

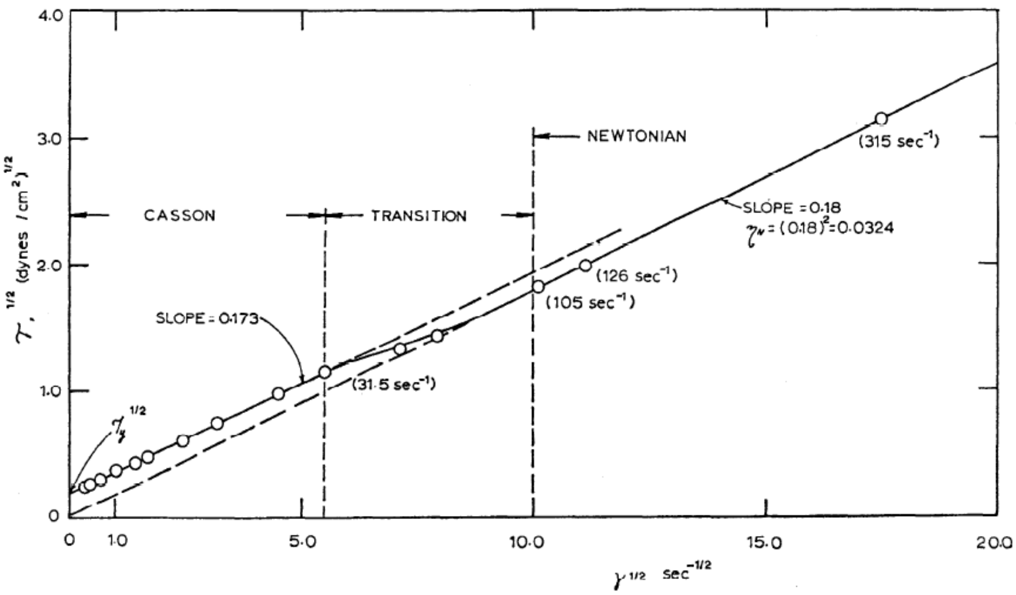


*Figure 1: Casson plot showing the relationship between the square root of the shear rate, $\dot{\gamma}$, and the square root of the shear stress, $\tau = \tau_{rz}$ for human blood [1]*

where $s$ is the slope of the line equal to the square root of the ultimate Newtonian viscosity and $\tau_y$ is the yield stress of the blood.

As the goal of this paper is to determine the velocity profile of blood in a cylindrical pipe, we must first solve the Casson representation of a non-Newtonian fluid for $\dot{\gamma} = -\frac{dv_z}{dr}$ and integrate with respect to $r$. We can therefore write:

$$\dot{\gamma} = -\frac{dv_z}{dr} = \frac{1}{s^2}\left(\tau_{rz}^{1/2} - \tau_y^{1/2}\right)^2 \tag{2}$$

In order to integrate this system, we need to find an expression for $\tau_{rz}$ such that $\tau_{rz} = \tau_{rz}(r)$. This is accomplished with a force balance on a cylindrical control volume within the system as shown in *Figure 2*. This provides us with the following equation:

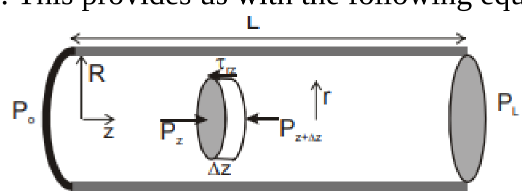


*Figure 2: Visual representation of the force balance for a control volume in our system [2]*

$$\pi r^2 P|_z - \pi r^2 P|_{z+\Delta z} - 2\pi r \Delta z \tau_{rz} = 0 \tag{3}$$

where after dividing by the volume, rearranging the equation, and taking the limit as $\Delta z$ approaches 0 yields:

$$\frac{2}{r}\tau_{rz} = -\frac{dP}{dz} \tag{4}$$

As the left side of the equation is only a function of radius, $r$, and the right side of the equation is only a function of $z$, both sides must equal a constant. As we have a known pressure differential over a known length, we can express this constant as $\frac{(P_o - P_L)}{2L}$ where $P_o$ and $P_L$ are the inlet and outlet pressures, respectively, and $L$ is the length of the cylinder. Now, $\tau_{rz}$ can be expressed as a function of $r$:

$$\tau_{rz} = \frac{(P_o - P_L)}{2L} r \tag{5}$$

Another important conclusion that will come from Eq. (4) is the maximum shear stress, $\tau_{\mathrm{w}}$, which is experienced at the wall of the cylinder. This can be written as:

$$\tau_{\mathrm{w}} = \frac{(P_o - P_L)}{2L} \mathrm{R} \tag{6}$$

where $R$ is the inner radius of the cylinder. Combining Eq. (5) and Eq. (6) together, we get the following relationship for $\tau_{rz} = \tau_{rz}(r)$ :

$$\tau_{rz} = \frac{\tau_w}{R} r \tag{6}$$

This can be inserted into Eq. (2) to give the expression:

$$\dot{\gamma} = -\frac{dv_z}{dr} = \frac{1}{s^2}\left[\frac{\tau_w}{R} r - 2\left(\frac{\tau_w}{R}\right)^{1/2} r^{1/2} \tau_y^{1/2} + \tau_y\right] \tag{7}$$

Integrating this equation and solving for $v_z$, it is shown that:

$$v_z(r) = \frac{R\tau_w}{2s^2}\left\{\left[1 - \left(\frac{r}{R}\right)^2\right] - \frac{8}{3}\left(\frac{\tau_y}{\tau_w}\right)^{1/2}\left[1 - \left(\frac{r}{R}\right)^{3/2}\right] + 2\left(\frac{\tau_y}{\tau_w}\right)\left(1 - \frac{r}{R}\right)\right\} \tag{8}$$

It should be noted that the integration bounds for $r$, and thus the domain of Eq. (8), are for $r_{crit} \leq r \leq R$ where $r_{crit}$ is the critical radius such that:

$$\tau_{rz} = \tau_y = -\frac{r_{crit}}{2}\frac{dP}{dz} \tag{9}$$

This constraint is made because at any point beneath, $r_{crit}$, the yield stress of the blood is greater than the shear stress and the velocity becomes constant. By combining Eq. (6) and Eq. (9), we can write:

$$\frac{r_{crit}}{R} = \frac{\tau_y}{\tau_w} \qquad \text{for } r \leq r_{crit} \tag{10}$$

This value can be plugged into Eq. (8) to get the constant velocity at the center of the pipe, $v_{core}$, for any $r \leq r_{crit}$ and can be expressed as:

$$v_{core}(r) = \frac{R\tau_w}{2s^2}\left\{\left[1 - \left(\frac{\tau_y}{\tau_w}\right)^2\right] - \frac{8}{3}\left(\frac{\tau_y}{\tau_w}\right)^{1/2}\left[1 - \left(\frac{\tau_y}{\tau_w}\right)^{3/2}\right] + 2\left(\frac{\tau_y}{\tau_w}\right)\left(1 - \frac{\tau_y}{\tau_w}\right)\right\} \tag{11}$$

In deriving the expressions for the velocity profile of blood, several assumptions were made. These include the length of the tube, $L$, being much greater than the radius of the tube, $R$ such that $L/R > 100$ to eliminate entrance effects. It is also assumed that we are operating in a steady, fully-developed system with isothermal flow where blood is incompressible, the no-slip boundary condition is applicable at the walls of the tube, and momentum transfer is one dimension in the $z$-direction only such that $v_z = v_z(r)$.

**Results and Discussion**

From Eq. (8) and Eq. (9) above, we are able to model the velocity profile of blood as shown in *Figure 3*. This plot was created using an average pressure drop of 9333 dynes/cm$^2$ (7mmHg) and a radius of 0.07 cm, both typical for smaller veins [3, 4]. A length of 1000 times the radius was used to guarantee the inexistence of entrance effects. Parameters varied by source for the yield stress, $\tau_y$, and the square root of the ultimate Newtonian viscosity, $s$. *Table 1* in Appendix I shows the values used for the constants for each curve below.

From *Figure 3*, it is observed that the flow profiles of blood are blunter than those of a Newtonian fluid under the same conditions (with a constant viscosity of blood used representative shear rate above 100 s$^{-1}$) [1]. This is due to the presence of yield stress in the blood, and we would expect a similar behavior in other Casson fluids. It is also observed that Merrill's values for $\tau_y$ and $s$ align better with a Newtonian flow profile than Nguyen's which

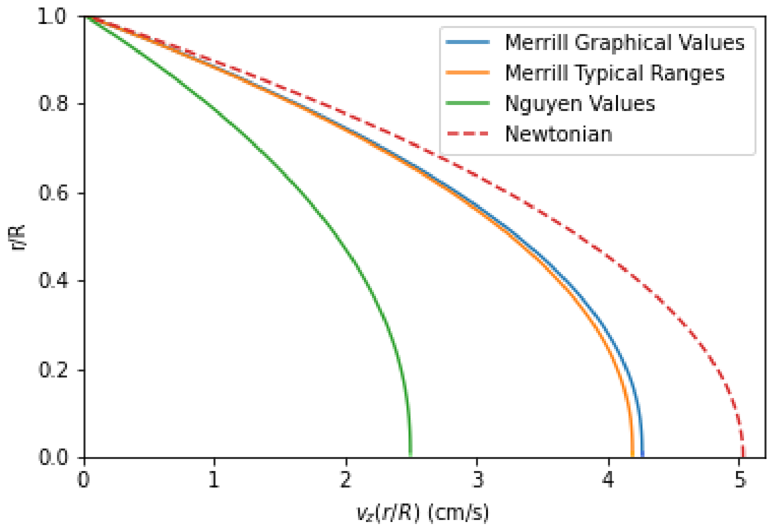


*Figure 3: Velocity profile of blood through a cylinder using Casson equation (solid lines) and comparing them to a Newtonian fluid under the same conditions (dashed line).*

reach just half of the maximum velocity of a Newtonian fluid. In fact, at lower velocities (< 1cm/s), Merrill's profiles almost correspond directly to Newtonian flow profiles.

Several of our assumptions become evident in the velocity profile shown in *Figure 3*. The no-slip boundary condition is what gives all four plots the intersection at coordinate (0, 1) of the graph. Likewise, though difficult to see on this particular graph, our assumptions of pressure and length affect the value of $r_{crit}$ where $v_z = v_{core}$ and the velocity profile is vertical. For our chosen pressure and length, $r_{crit}$ is on the scale of micrometers and it is difficult to see on the graph. To see the effect of pressure and length on $r_{crit}$, please see *Figure 4* in Appendix I. It is interesting to note that though we have made an average pressure assumption, the values $r_{crit}$ should be constantly changing with every heartbeat due to the fluctuating pressure differential.

**Conclusion**

Merrill's paper was the first paper to break blood into its components and successfully determine what gives it its non-Newtonian nature, including the behavior of blood for many different blood diseases, such as anemia. After this paper's publication, many papers discussing the rheological conditions of blood and how differing factors in a patient's blood (such as the hematocrit level, fibrinogen level, etc.) might change the flow pattern and, therefore, the method to treat and prevent different diseases, such as thrombosis. Several studies have also been done linking the rheology of blood to personal fitness and health [2].

The applications of a Casson fluid flow profile extend to much more than blood, however. Other Casson fluids include pharmaceuticals, jelly, tomato sauce, and concentrated fruit juices [5]. An understanding of these flow profiles can also help in wide-scale manufacturing where these fluids are transported through pipes into their respective containers.

It was incredible being able to learn as I went through this derivation. Through this process, I learned how a force balance is really just a momentum balance, how the yield stress is able to affect flow, and how one combine force balances and non-Newtonian models to accurately model non-Newtonian fluids. It was also incredibly interesting to me to learn the reasoning behind the transport properties of blood. Thank you for this opportunity to apply the transport phenomena principles we learned in class in a new and insightful way.

References

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Appendix I

*Table 1: Values of $\tau_y$ and $s$ used to produce Figure 3 [1, 2]*

| *Model* | $\tau_y$ | $s$ |
|---|---|---|
| Merrill Graphical Values | 0.035 | $0.03^{1/2}$ |
| Merrill Typical Ranges | 0.04 | $0.03^{1/2}$ |
| Nguyen Values | 0.0289 | 0.229 |
| Newtonian | 0 | $0.0324^{1/2}$ |

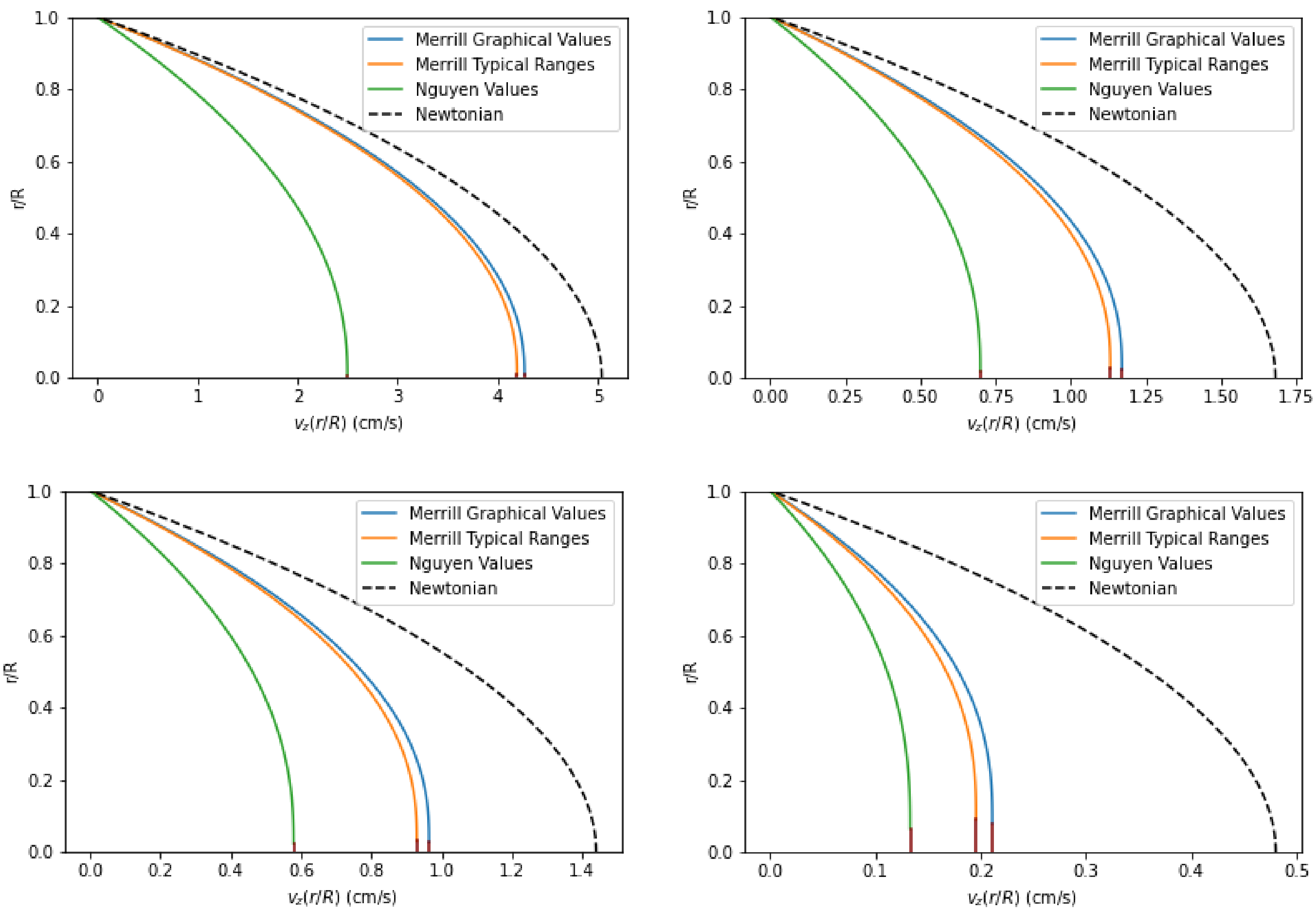


*Figure 4: Different pressure and temperature conditions to see how $r_{crit}$ changes. Vertical flow profile influenced by $r_{crit}$ indicated in red on the graphs. Top Left: ΔP = 7mmHg, L = R*1000; Top Right: ΔP = 7mmHg, L = R*3000; Bottom Left: ΔP = 2mmHg, L = R*1000; Bottom Right: ΔP = 2mmHg, L = R*3000.*