Integral Forms of the energy equation

See Panton § 7.2, 5.9–5.10, $\underbrace{3.12\text{–}3.13}_{\text{GDT, LIR}}$

&

Deen § 11.4

Total energy equation

$$\frac{\partial}{\partial t}\left[\rho\left(e+\tfrac{1}{2}v^2\right)\right] + \nabla\cdot\left[\rho\underline{v}\left(e+\tfrac{1}{2}v^2\right)\right] = -\underline{\nabla}\cdot\underline{q} + \underline{\nabla}\cdot\left(\underline{\underline{T}}\cdot\underline{v}\right) + \rho\underline{v}\cdot\underline{F}$$

(Panton notation ↑)

$\underline{F} = \nabla(gz) = \underline{g}$

$$\frac{\partial}{\partial t}\left[\rho\left(e+\tfrac{1}{2}v^2\right)\right] + \nabla\cdot\left[\rho\underline{v}\left(e+\tfrac{1}{2}v^2\right)\right] = -\nabla\cdot\underline{q} + \nabla\cdot\left(\underline{\underline{\sigma}}\cdot\underline{v}\right) + \rho\underline{v}\cdot\underline{g}$$

(Deen notation ↑)

$\underbrace{\nabla\cdot\left[\rho\underline{v}\left(e+\tfrac{1}{2}v^2\right)\right]}_{\text{rate of change of energy}}$

net heat flow ($-\nabla\cdot\underline{q}$); work by surface forces ($\nabla\cdot(\underline{\underline{\sigma}}\cdot\underline{v})$); work by body forces ($\rho\underline{v}\cdot\underline{g}$)

Mechanical Energy equation

$$\underbrace{\frac{\partial}{\partial t}\left(\rho\tfrac{1}{2}v^2\right) + \nabla\cdot\left(\rho\underline{v}\tfrac{1}{2}v^2\right)}_{\text{rate of change of kinetic energy}} = -\underline{v}\cdot\nabla p + \underline{v}\cdot\left(\underline{\nabla}\cdot\underline{\underline{\tau}}\right) + \rho\underline{v}\cdot\underline{F}$$

$-\underline{v}\cdot\nabla p$: pressure gradient accelerating fluid

$\underline{v}\cdot(\underline{\nabla}\cdot\underline{\underline{\tau}})$: stress gradient accelerating fluid.

$\rho\underline{v}\cdot\underline{F}$: gravitational potential accelerating fluid

$\underline{F} = \nabla\phi = \nabla(gz)$
$= \underline{g}$

Thermal Energy equation

$$\underbrace{\frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho\underline{v}e)}_{\text{rate of change of internal energy}} = -p\,\underline{\nabla}\cdot\underline{v} + \underline{\underline{\tau}}:\underline{\nabla}\underline{v} - \underline{\nabla}\cdot\underline{q}$$

$-p\,\underline{\nabla}\cdot\underline{v}$: work of expansion/contraction

$\underline{\underline{\tau}}:\underline{\nabla}\underline{v}$: viscous dissipation

$-\underline{\nabla}\cdot\underline{q}$: heat flux

Total = Mech + Thermal

(*) $\underline{\nabla}\cdot\left(\underline{\underline{T}}\cdot\underline{v}\right) = \underline{\nabla}\cdot\left(\underline{\underline{\sigma}}\cdot\underline{v}\right)$

$$= \nabla\cdot\left[\left(-p\underline{\underline{\delta}} + \underline{\underline{\tau}}\right)\cdot\underline{v}\right]$$

$$= -\nabla\cdot(p\underline{v}) + \nabla\cdot\left(\underline{\underline{\tau}}\cdot\underline{v}\right)$$

$$= \underset{(1)}{-p(\underline{\nabla}\cdot\underline{v})} - \underset{(2)}{\underline{v}\cdot\underline{\nabla}p} + \underset{(3)}{\underline{\underline{\tau}}:\nabla\underline{v}} + \underset{(4)}{\underline{v}\cdot(\underline{\nabla}\cdot\underline{\underline{\tau}})}$$

Thermal energy eq.

Mech. Energy Eq.

Re-arrange Mech. Energy equation

$$\frac{\partial}{\partial t}\left(\rho\tfrac{1}{2}v^2\right) + \nabla\cdot\left(\rho\underline{v}\tfrac{1}{2}v^2\right) = \underbrace{-\underline{v}\cdot\nabla p + \underline{v}\cdot(\underline{\nabla}\cdot\underline{\underline{\tau}})}_{\underline{v}\cdot(\underline{\nabla}\cdot\underline{\underline{\sigma}})} + \underbrace{\rho\underline{v}\cdot\underline{F}}_{\rho\underline{v}\cdot\nabla(gz)}$$

$$\rho\underline{v}\cdot\underline{\nabla}\phi \qquad \phi = gz$$

note that

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v}) = 0 \quad \leftarrow \text{continuity}$$

$$\phi\frac{\partial\rho}{\partial t} + \phi\underline{\nabla}\cdot(\rho\underline{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho\phi) + \phi\,\nabla\cdot(\rho\underline{v}) = 0$$

so:

$$\rho\underline{v}\cdot\nabla\phi = \rho\underline{v}\cdot\nabla\phi + \frac{\partial}{\partial t}(\rho\phi) + \phi\,\underline{\nabla}\cdot(\rho\underline{v})$$

$$= \frac{\partial}{\partial t}(\rho\phi) + \underbrace{\rho\underline{v}\cdot\nabla\phi + \phi\,\underline{\nabla}\cdot(\rho\underline{v})}_{\text{product rule}}$$

$$= \frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\underline{v}\phi)$$

so,

kinetic + potential

work done by stress gradients

$$\frac{\partial}{\partial t}\left(\rho \tfrac{1}{2} v^2 + \rho\phi\right) + \nabla\cdot\left(\rho \underline{v} \tfrac{1}{2} v^2 + \rho \underline{v}\phi\right) = \underline{v}\cdot(\underline{\nabla}\cdot\underline{\underline{\sigma}})$$

$$= \nabla\cdot(\underline{\underline{\sigma}}\cdot\underline{v}) - \underline{\underline{\sigma}} : \nabla\underline{v}$$

work done by surface forces − viscous / expansionary dissipation

Integral Forms

* Remember:

GDT: $\displaystyle\int_V \underline{\nabla}\cdot\underline{f}\, dV = \int_S \underline{f}\cdot\underline{n}\, dS$

LIR: $\displaystyle\frac{d}{dt}\int_{V(t)} f\, dV = \int_V \frac{\partial f}{\partial t}\, dV + \int_S \underline{n}\cdot\underline{u}\, f\, dS$

$$\frac{d}{dt}\int_V \left[\rho\tfrac{1}{2}v^2 + \rho\phi\right] dV - \int_S \underline{n}\cdot(\underline{v}-\underline{u})\left[\rho\tfrac{1}{2}v^2 + \rho\phi\right] dS =$$

$$+\int_S \underline{n}\cdot\underline{\underline{\sigma}}\cdot\underline{v}\, dS - \int_V \sigma : \nabla v\, dV$$

$$\frac{d}{dt}\int_V \rho\left[\tfrac{1}{2}v^2 + \phi\right] dV - \int_S \rho\,\underline{n}\cdot(\underline{v}-\underline{u})\left[\tfrac{1}{2}v^2 + \phi\right] dS =$$

$$-\int_S p\,\underline{n}\cdot\underline{v}\, dS + \int_S \underline{n}\cdot\underline{\underline{\tau}}\cdot\underline{v}\, dS + \int_V (p\,\underline{\nabla}\cdot\underline{v})\, dV$$

E_c: rate of loss due to compression/expansion

$$-\int_V (\tau : \nabla v)\, dV$$

loss due to viscous friction.