Lecture 26 - Integral & Engineering Momentum Balance

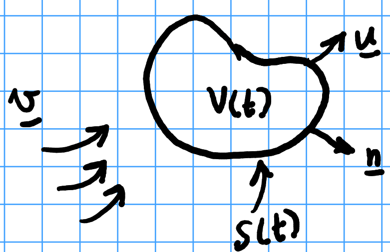
* Reading: Deen 11.3
I. Integral momentum balance

II. Engineering momentum balance

III. Examples

I. Integral momentum balance

* Last time we derived an integral mass balance for an arbitrary control volume



V(t): C.V.

S(t): C.S.

$\underline{v}$: fluid vel.

$\underline{u}$: velocity of C.S.

$\underline{n}$: unit normal for C.S.

$$\underbrace{\frac{d}{dt}\int_{V(t)} \rho \, dV}_{\text{accum.}} + \underbrace{\int_{S(t)} \rho (\underline{v}-\underline{u}) \cdot \underline{n} \, dS}_{\text{out-in}} = \underbrace{0}_{\text{gen.}}$$

* We can generalize this balance for any property, B.

B: property of interest (mass, ex.)

b: property per unit volume (e.g. density)

$$\frac{d}{dt}\int_{V(t)} b \, dV + \int_{S(t)} \underbrace{b (\underline{v}-\underline{u}) \cdot \underline{n}}_{\text{conv. flux of b}} \, dS = \underbrace{\dot{B}_{gen}}_{\text{generation of b}}$$

* "general integral property balance"
  ↳ Reynold's transport theorem

* Momentum : $\underline{B} = m\underline{v}$
$\underline{b} = \rho \underline{v}$

momentum gen. & diffusive fluxes

$$\frac{d}{dt}\int_{V(t)} \rho \underline{v}\, dV + \int_{S(t)} \rho \underline{v}(\underline{v}-\underline{u})\cdot \underline{n}\, dS = \dot{\underline{B}}_{gen}$$

Newton's 2nd law

$$\dot{\underline{b}}_{gen} = \sum_i \underline{F}_i = \int_{V(t)} \underline{f}_{body}\, dV + \int_{S(t)} \underline{s}_{surfaces}\, dS$$

$$= \int_{V(t)} \rho \underline{g}\, dV + \int_{S(t)} \underline{n}\cdot\underline{\underline{\sigma}}\, dS$$

$$= \int_{V(t)} \rho \underline{g}\, dV + \int_{S(t)} -\underline{n}P\, dS + \int_{S(t)} \underline{n}\cdot\underline{\underline{\tau}}\, dS$$

* the final Equation

$$\boxed{\frac{d}{dt}\int_{V(t)} \rho \underline{v}\, dV + \int_{S(t)} \rho \underline{v}(\underline{v}-\underline{u})\cdot \underline{n}\, dS = \int_{V(t)} \rho \underline{g}\, dV - \int_{S(t)} \underline{n}P\, dS + \int_{S(t)} \underline{n}\cdot\underline{\underline{\tau}}\, dS}$$

Integral momentum balance. ← more general than Newton's 2nd law.

II. Engineering momentum balance

* The integral balance always applies, but it isn't very useful.
* We want to pick a "smart" control volume.

  - discrete inlets & outlets
  - uniform density @ I/O

- unidirectional flow @ I/O (on average) turbulent ✓
- small viscous stresses @ I/O.

  - pipes, in & out
  - pumps, turbulent wakes, turbines, fans
    sudden contractions / expansions

- fixed control volume.

⇒ engineering momentum balance. (see extra notes)

p. 296-298 in Deen for math

$$\boxed{\frac{d}{dt}(m\underline{v})_{cv} = \sum_i^{in} w_i a_i \underline{v}_i - \sum_i^{out} w_i a_i \underline{v}_i + \sum_i F_i}$$

mass c.v   mass flow   velocity   pressure, $m\underline{g}$, etc

$$a_i = \frac{\langle v_i^2 \rangle}{\langle v_i \rangle^2}$$

$$= \begin{cases} 1, & \text{plug flow} \\ \frac{50}{49}, & \text{turbulent} \approx 1 \\ \frac{4}{3}, & \text{laminar flow} \end{cases}$$

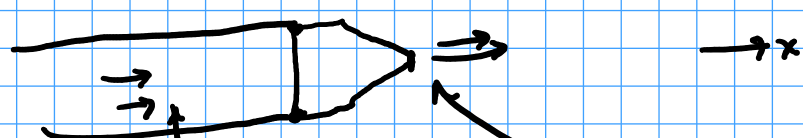
$\langle \underline{v} \cdot \underline{v} \rangle = \langle v^2 \rangle$

$\langle \underline{v} \rangle \cdot \langle \underline{v} \rangle = \langle v \rangle^2$

III. Examples

A. Flow in a nozzle, c.v. 1

* consider flow in a hose w/ a nozzle

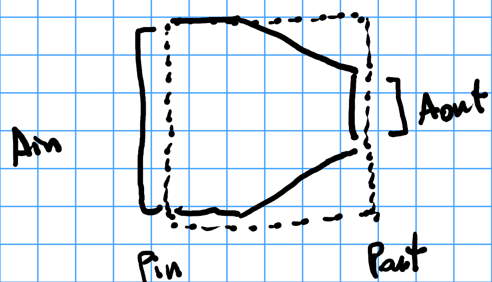


- assume turbulent

P_{in}, v_{in}, A_{in} $\quad$ $P_{out} = P_{atm}, v_{out}, A_{out}$

• what is the force of the nozzle needed to keep the hose from moving?

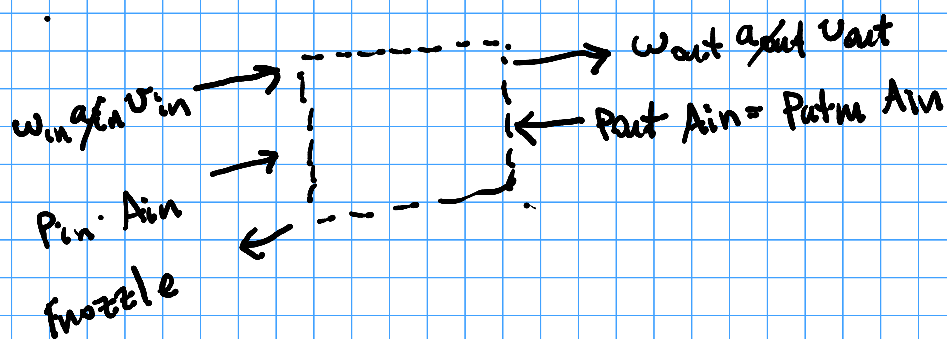
(1) Draw a control volume.



- discrete I/O ✓
- uniform density @ I/O ✓
- unidirectional flow @ I/O ✓
- small viscous stress @ I/O ✓
- fixed C.V. ✓

(2) write balance

$\alpha_{in} = \alpha_{out} = \frac{50}{49} \approx 1$



$$0 = w_{in} v_{in} - w_{out} v_{out} + P_{in} A_{in} - P_{atm} A_{in} - F_{nozzle}$$

(3) Math

$$F_{nozzle} = w_{in} v_{in} - w_{out} v_{out} + A_{in}(P_{in} - P_{atm})$$

$$w_{in} = w_{out} = \rho A_{in} v_{in} = \rho A_{out} v_{out}$$

$$v_{out} = v_{in} \frac{A_{in}}{A_{out}}$$

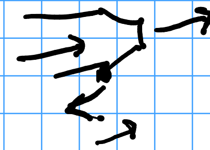
$$= \rho A_{in} v_{in}^2 - \rho A_{in} v_{in}^2 \frac{A_{in}}{A_{out}} + A_{in} P_{in,gauge}$$

$$\boxed{F_{nozzle} = A_{in}\left[\rho v_{in}^2\left(1-\frac{A_{in}}{A_{out}}\right) + P_{in,gauge}\right]}$$

inertia

$A_{out} < A_{in}$

force for p

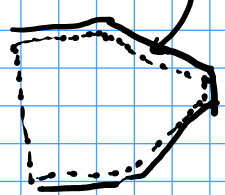


B. Flow in a nozzle, C.V 2

* A new C.V. that gives the same answer.

(1) Draw C.V.

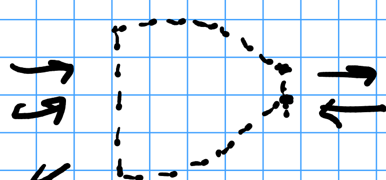
inside nozzle



• discrete I/O V

$\alpha_{in} \approx \alpha_{out} \approx 1$

(2) write balance



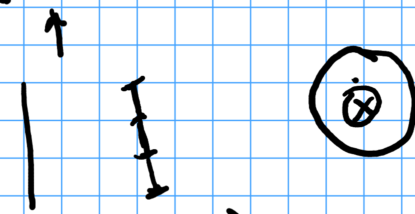
$$0 = w_{in} v_{in} - w_{out} v_{out} + P_{in} A_{in} - P_{out} A_{out} - F_{nozzle}$$

(3) Math

: (Do on your own)

$$F_{nozzle} = A_{in}\left[P_{in} - P_{out}\cdot\frac{A_{out}}{A_{in}} + \rho v_{in}^2\left(1-\frac{A_{in}}{A_{out}}\right)\right]$$

$$-F_{net,\,nozzle} = -F_{nozzle} + P_{atm}(A_{in} - A_{out})$$



$$F_{nozzle,\,net} = F_{nozzle} - P_{atm}(A_{in} - A_{out})$$

$$= P_{in}A_{in} - \cancel{P_{out}}A_{out} + \rho v_{in}^2 A_{in}\left(1 - \frac{A_{in}}{A_{out}}\right)$$

$$- \underbrace{P_{atm}A_{in}}_{P_{in,gauge}\cdot A_{in}} + \cancel{P_{atm}}A_{out} \qquad P_{out} = P_{atm}$$

$$= A_{in}\left[P_{in,\,gauge} + \rho v_{in}^2\left(1 - \frac{A_{in}}{A_{out}}\right)\right]$$

same as (A)