Lecture 22 – Boundary Layer Theory

* Reading: Deen 9.3–9.5

I. D'Alembert's paradox

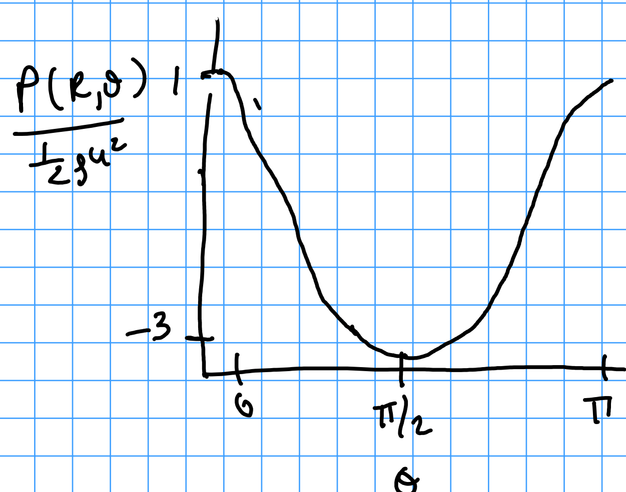
II. Boundary Layers

A. BL Equations & Solutions

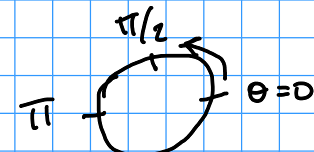
B. Calc. Drag Coeffs

I. D'Alembert's paradox

* Drag for inviscid & irrotational $\Rightarrow F_D = 0$
* What is the problem?
  - No friction drag: ok – high Re, neglected μ
  - No pressure drag! $\frac{P(R,\theta)}{\frac{1}{2}\rho u^2} = 1 - 4\sin^2\theta$



Press: symmetric

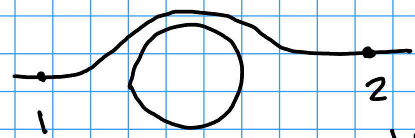


* Why is $P(R,\theta)$ symmetric?
  - Inviscid flow obeys Bernoulli's Eq.

Energy Bal on a streamline

$$\frac{P}{\rho} + \frac{v^2}{2} = \text{const} \quad \text{on a streamline}$$

$$\frac{\Delta p}{\rho} + \frac{\Delta v^2}{2} = 0$$

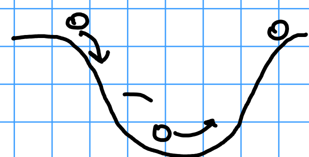


$u \rightarrow$ $v_1 = v_2$ $u \rightarrow$

$p_1 = p_2 \Rightarrow$ no pressure drag

* To break symmetry → I still need friction

$$\frac{\Delta p}{\rho} + \frac{\Delta v^2}{2} = \underbrace{-E_v}_{} \leftarrow \text{friction, energy loss}$$



p = potential energy

v = kinetic "

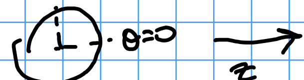
II. Boundary Layers

* How do I add friction back in? By B.L.

* Start over w/ Navier-Stokes?

* A clue:

$\theta = \pi/2 \rightarrow \underline{e}_\theta = -\underline{e}_z$

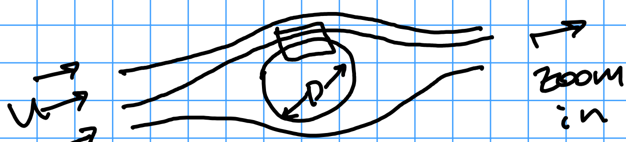


$$v_\theta = -u \underbrace{\sin\theta}_{1} \Big[1 + \underbrace{\left(\frac{R}{r}\right)^3}_{2} \Big]$$

$$v_\theta(R, \pi/2) = -2u$$

doesn't satisfy no-slip! → $v_z = 2u$

* what if I have 2 regions!?



outer region

* inviscid flow
* char. size: D

$$Re = \frac{\rho u D}{\mu} \gg 1$$

* Re is big — ok

Euler

$\underline{\omega} = 0 \rightarrow$ Laplace

inner region

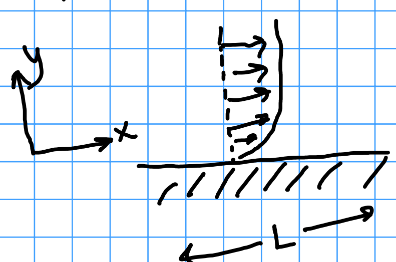
* no slip satisfied
* char size: δ

δ: distance to go for $v = 0$ to $v = 2u$

$$1 \approx Re_\delta = \frac{\rho u \delta}{\mu} = \frac{\rho u^2}{\mu u/\delta}$$

* if δ is small then Re isn't big! Viscosity will matter! we will get friction back!

A. Boundary Layer Equations & Solutions



* Assumptions ← δ is small
  - Flat surface (cartesian) $\delta \ll L$, large radius of curvature
  - 2D
  - steady

similar non-dimensionalization of N.S.

$$Re_L = \frac{\rho u L}{\mu} \gg 1 \quad , \quad \frac{\delta}{L} \ll 1 \longrightarrow \text{B.L Equations}$$

*supp. notes on this derivation (not too bad)

* Boundary Layer Equations

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 v_x}{\partial y^2}$$

* like creeping flow, inviscid flow,
  we get this by dimensional analysis
  → PDE → You don't have to solve (out of scope)

Solution 1: Blasius' exact solution

- Ex. 9.3-1 on pp. 239-242 in Deen
- No formula for v_x → "no closed form solution"
  ↳ computer only
- we're going to ignore this soln.

Solution 2: Von-Kármán – Pohlhausen approx. soln.

$$v_x = U\left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]$$

$$\delta(x) = \sqrt{\frac{840}{39}\frac{\nu x}{U}}$$

ν: kinematic viscosity

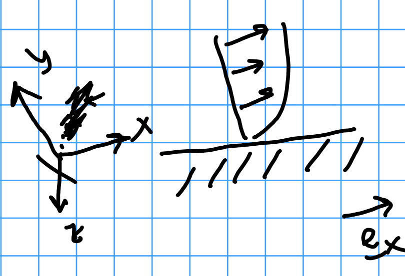
$\nu = \mu/\rho$

$P = \text{const}$

$v_y = 0$

- Not as accurate
- Not in your book ☹ → Panton § 20.4 pp. 506-508
  Bird, Stewart & Lightfoot (BSL) § 4.4 pp. 136-137

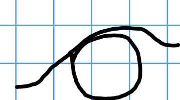
B. Calculate Drag Coefficient



$c_f = ?$

$x: [0, L]$

$z: [0, W]$



$c_D = \frac{24}{Re}$

$c_D = 0.445?$

wake + Turb B.L.?

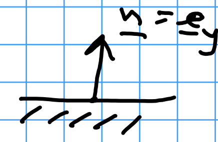
(1) Get solution

$$v_x = U\left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] \qquad \delta = \sqrt{\frac{840}{39}\,\frac{\nu x}{U}}$$

$$v_y = 0$$

$$p = \text{const}$$

(2) simplify drag force formula



$$F_D = -\int_A \underbrace{\cancel{\underline{n}\cdot\underline{e}_x}}_{0}\, P\, dA + \int_A \underline{n}\cdot\underline{\underline{\tau}}\cdot\underline{e}_x\, dA$$

$$\underline{n}\cdot\underline{e}_x = \underline{e}_y\cdot\underline{e}_x = 0$$

$$\underline{n}\cdot\underline{\underline{\tau}}\cdot\underline{e}_x = \underline{e}_y\cdot\underline{\underline{\tau}}\cdot\underline{e}_x = \tau_{yx}$$

$$\tau_{yx} = \mu\left[\frac{\partial v_x}{\partial y} + \underbrace{\frac{\partial v_y}{\partial x}}_{0}\right] = \mu\frac{\partial v_x}{\partial y}$$

$$F_D = \int_0^W\int_0^L \mu\left.\frac{\partial v_x}{\partial y}\right|_{y=0} dx\, dz$$

(3) Find τ_{yx}, taking derivatives

$$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\left\{U\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\right\}$$

$$= U\left[\frac{3}{2}\frac{1}{\delta} - \frac{1}{2}\,3\left(\frac{y}{\delta}\right)^2\frac{1}{\delta}\right]$$

$$= \frac{3}{2}\frac{U}{\delta}\left[1 - \left(\frac{y}{\delta}\right)^2\right]$$

$$\tau_{yx}\Big|_{y=0} = \frac{3\mu U}{2\delta}\left(1 - \cancel{\left(\frac{0}{\delta}\right)^2}\right) = \frac{3\mu U}{2\delta}$$

(4) Evaluate Integral

$$F_D = \int_0^W \int_0^L \frac{3\mu U}{2\delta(x)}\, dx\, dz \qquad \delta = \sqrt{\frac{840}{39}\frac{\nu x}{U}}$$

$$= \frac{3\mu U}{2}\sqrt{\frac{39}{840}} \int_0^W \int_0^L \left(\frac{\nu x}{U}\right)^{-1/2} dx\, dz$$

$$= \frac{3\mu U W}{2}\sqrt{\frac{39}{840}} \sqrt{\frac{U}{\nu}} \left(2x^{1/2}\right)\Big|_0^L$$

$$F_D = \frac{3}{2}\mu U W \sqrt{\frac{39}{840}} \sqrt{\frac{UL}{\nu}} \quad ✓$$

$$C_f = \frac{F_D / A_{11}}{\frac{1}{2}\rho U^2} = \frac{3}{\cancel{2}}\mu \cancel{U} \cancel{W} \sqrt{\frac{39}{840}} \sqrt{\frac{UL}{\nu}} \cdot \frac{1}{L\cancel{W}} \frac{\cancel{2}}{\rho U^{\cancel{2}}}$$

$$= \frac{3\mu}{L\rho U}\sqrt{\frac{39}{840}} \sqrt{\frac{UL\rho}{\mu}} = \frac{3\sqrt{39}}{\sqrt{840}} \frac{1}{\underset{Re^{1/2}}{\cancel{Re}}} \cancel{\sqrt{Re}}$$

$$\boxed{C_f = \frac{3\sqrt{39}}{\sqrt{840}} Re^{-1/2} \approx 1.292\, Re^{-1/2}}$$

Blasius

7.3-18 } Deen
3.2-15

laminar BL