Lecture 21 - Drag Force Calculations

I. Drag force formula

II. Drag force calculations

A. Ex: Drag on a sphere in creeping flow

B. Ex " " " ^ inviscid flow

* Deen: 6.6, 8.3, T.2 ← p. 152-154

I. Drag force formula

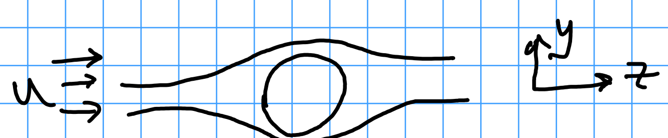
$$\underline{F} = A\,\underline{n}\cdot\sigma \rightarrow d\underline{F} = \underline{n}\cdot\underline{\underline{\sigma}}\,dA$$

$$\int d\underline{F} = \underline{F} = \int_A \underline{n}\cdot\underline{\underline{\sigma}}\,dA$$

* remember: $\underline{\underline{\sigma}} = -\underline{\underline{\delta}}\,P + \underline{\underline{\tau}}$

$$\underline{F} = -\int_A \underbrace{\underline{n}\cdot\underline{\underline{\delta}}}\,P\,dA + \int_A \underline{n}\cdot\underline{\underline{\tau}}\,dA$$

$$\boxed{\underline{F} = -\int_A \underline{n}\,P\,dA + \int_A \underline{n}\cdot\underline{\underline{\tau}}\,dA}$$

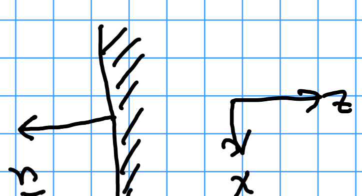


$\underline{F}\cdot\underline{e}_z$: drag

$\underline{F}\cdot\underline{e}_y$: lift

$$F_D = \underline{F}\cdot\underline{e}_z = \underbrace{-\int_A \underline{n}\cdot\underline{e}_z\,P\,dA}_{\text{pressure drag}} + \underbrace{\int \underline{n}\cdot\underline{\underline{\tau}}\cdot\underline{e}_z\,dA}_{\text{friction drag}} \quad *$$

Ex: simplify the drag force formula



(y: into the page)

width: W

$$F_D = -\int_0^L\int_0^W \underline{n}\cdot\underline{e}_z\, P(x,y)\, dx\, dy \quad (a)$$

$$+\int_0^L\int_0^W \underline{n}\cdot\underline{\underline{\tau}}(x,y)\cdot\underline{e}_z\, dx\,dy \quad (b)$$

$$\underline{n} = -\underline{e}_z$$

(a) $\underline{n}\cdot\underline{e}_z = -\underline{e}_z\cdot(\underline{e}_z) = -1$

(b) $\underline{n}\cdot\underline{\underline{\tau}}\cdot\underline{e}_z = -\underline{e}_z\cdot\underline{\underline{\tau}}\cdot\underline{e}_z = -\tau_{zz}$

$$-[0\ 0\ 1]\begin{bmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$-\begin{bmatrix}\tau_{zx} & \tau_{zy} & \tau_{zz}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = -\tau_{zz}$$

$$F_D = \int_0^L\int_0^W P(x,y)\, dx\, dy - \int_0^L\int_0^W \tau_{zz}(x,y)\, dx\, dy$$

II. Drag Force Calcs

* steps:

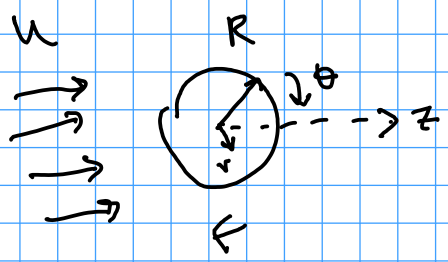
(1) Need a solution for $\underline{v}$ & P (in book, given, etc.) ← PDEs

(2) simplify drag force formula for coord sys.

(3) Find parts of τ ← velocity derivatives

(4) Do integrals

A. Example: Drag on a sphere in creeping flow $(Re \ll 1)$



$$0 = -\nabla P + \mu \nabla^2 v$$
$$0 = \nabla \cdot v$$

* creeping flow

* axisymmetric

$v_\phi = 0$

(1) Solution: Ex. 8.3-2 in Deen p.201-204

$$v_r(r,\theta) = U\left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta$$

$$v_\theta(r,\theta) = -U\left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta$$

$$P(r,\theta) = -\frac{3}{2}\frac{\mu U}{R}\left(\frac{R}{r}\right)^2\cos\theta$$

(2) Drag force formula

* spherical coords: Eq. 8.3-38 p.205

solid sphere

$$F_D = -2\pi R^2 \int_0^\pi \left[P(R,\theta)\cos\theta + \tau_{r\theta}(R,\theta)\sin\theta\right]\sin\theta \, d\theta$$

(3) Get $P(R,\theta)$, $\tau_{r\theta}(R,\theta)$

$$P(R,\theta) = -\frac{3}{2}\frac{\mu U}{R}\cos\theta$$

$$\tau_{r\theta} = \mu\left[\underbrace{r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)}_{(i)} + \underbrace{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}_{(ii)}\right]$$

table 6.6 p.143

(i) $$r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = r\frac{\partial}{\partial r}\left(-U\left[\frac{1}{r} - \frac{3R}{4r^2} - \frac{R^3}{4r^4}\right]\sin\theta\right)$$

$$= r(-U\sin\theta)\left[-\frac{1}{r^2} - (-2)\frac{3R}{4r^3} - (-4)\frac{R^3}{4r^5}\right]$$

$$= -U\sin\theta\left[-\frac{1}{r} + \frac{3R}{2r^2} + \frac{R^3}{r^4}\right]$$

(ii) $$\frac{1}{r}\frac{\partial u_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]U\cos\theta\right\}$$

$$= -U\sin\theta\left\{\frac{1}{r} - \frac{3R}{2r^2} + \frac{R^3}{2r^4}\right\}$$

$$\tau_{r\theta} = -\mu U\sin\theta\left[-\cancel{\frac{1}{r}} + \cancel{\frac{3R}{2r^2}} + \frac{R^3}{r^4} + \cancel{\frac{1}{r}} - \cancel{\frac{3R}{2r^2}} + \frac{R^3}{2r^4}\right]$$

$$\tau_{r\theta} = -\frac{3}{2}\mu U\sin\theta\frac{R^3}{r^4}$$

$$\tau_{r\theta}(R,\theta) = -\frac{3}{2}\frac{\mu U\sin\theta}{R}$$

(4) Eval integrals

- pressure drag:

$$F_{D,p} = -2\pi R^{2} \int_0^{\pi} -\frac{3}{2}\frac{\mu U}{R}\cos^2\theta \sin\theta \, d\theta$$

$$= -3\pi R \mu U \underbrace{\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta}_{2/3}$$

$$\boxed{F_{D,p} = -2\pi \mu U R}$$

• Friction drag

$$F_{D,\tau} = -2\pi R^{2} \int_0^{\pi} -\frac{3}{2}\frac{\mu U}{R} \sin^3\theta \, d\theta$$

$$= -3\pi \mu U R \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3}$$

$$\boxed{F_{D,\tau} = -4\pi \mu U R}$$

$$\boxed{F_D = -6\pi \mu U R}$$

$C_D = ?$

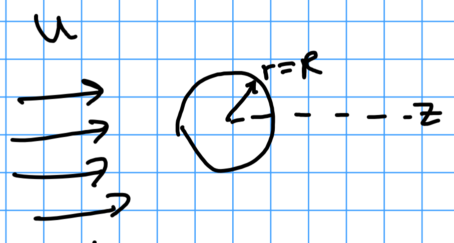
$$C_D = \frac{|F_D| / A_\perp}{\frac{1}{2}\rho U^2} = \frac{6\pi \mu U R}{\pi R^2} \frac{2}{\rho U^2} = \frac{12\mu}{\rho U R}$$

$$= \frac{24\mu}{\rho U D}$$

$$\boxed{C_D = \frac{24}{Re}}$$

We did it! Laminar flow

B. __Example:__ Drag on a sphere in inviscid flow



$\nabla^2\phi = 0$ * potential flow

$\underline{v} = \nabla\phi$ * axisymmetric

$\underline{\omega} = 0$ Inviscid & irrotational

(1) Solution: Ex 9.2-1 is for a __cylinder__

__My notes__ are for a sphere.

$$v_r = U\cos\theta\left[1 - \left(\frac{R}{r}\right)^3\right]$$

$$v_\theta = -U\sin\theta\left[1 + \left(\frac{R}{r}\right)^3\right]$$

$$P(r,\theta) = \frac{1}{2}\rho U^2\left(\frac{R}{r}\right)^3\left[2 - 4\sin^2\theta - \left(\frac{R}{r}\right)^3\right]$$

(2) Drag force formula

$$\tau_{r\theta} = \underset{0}{\cancel{\mu}}[\;]$$

$$F_D = -2\pi R^2\int_0^\pi P(R,\theta)\cos\theta\sin\theta\, d\theta$$

(3) Get $P(R,\theta)$ __no τ__!

$$P(R,\theta) = \rho U^2\left[1 - 4\sin^2\theta\right]$$

(4) Do integrals

$$F_D = -2\pi R^2 \int_0^{\pi} \rho u^2 \left[1 - 4\sin^2\theta\right] \cos\theta \sin\theta \, d\theta$$

$$= -2\pi R^2 \rho u^2 \underbrace{\int_0^{\pi} (1 - 4\sin^2\theta) \cos\theta \sin\theta \, d\theta}_{0}$$

$$\boxed{F_D = 0}$$

* what?! $C_D = 0.445$?

what is going on?

* This is called d'Alembert's paradox

wait until next time!!