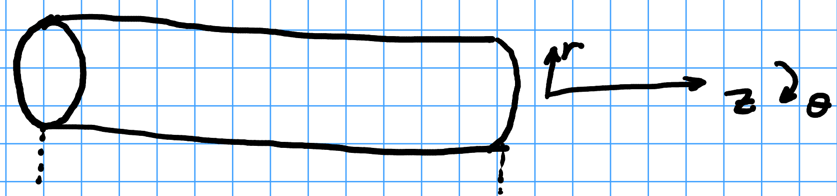
Lecture 19 – Non-Newtonian Pipe Flow

→ Flow profile for a power law fluid.

* See also example 7.5-1 in Deen, p. 175–176

Note the mistake in Eq. 7.5-8 → missing a factor $\frac{n}{n+1}$



$\mathcal{P} = P + \rho g z$

$z = 0$ $\quad$ $z = L$

$\mathcal{P}_0$ $\quad$ $\mathcal{P}_L$

$\tau = m s^n$ $\quad$ $\tau = m(2\underline{\underline{\Gamma}})^n$ $\quad$ $\|\underline{\underline{\Gamma}}\|$

$\underline{\underline{\tau}}$ $\quad$ $\underline{\underline{\Gamma}}$

Assume

- Steady → $\frac{\partial}{\partial t} \to 0$
- Axisymmetric → $\frac{\partial}{\partial \theta} \to 0$
- Unidirectional (v_z only) $v_\theta = v_r = 0$

$P + \rho g z$

$$\rho\left[\underbrace{\frac{\partial v_z}{\partial t}}_{0} + \underbrace{v_r \frac{\partial v_z}{\partial r}}_{0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{0} + \underbrace{v_z \frac{\partial v_z}{\partial z}}_{0}\right] = -\frac{\partial \mathcal{P}}{\partial z} +$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \underbrace{\frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{\theta z})}_{0} + \underbrace{\frac{\partial \tau_{zz}}{\partial z}}_{0}\right]$$

$\underline{\nabla} \cdot \underline{\underline{\tau}}$ $\quad$ $\tau = m s^n$?!

$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{bmatrix}$$

$\underline{\nabla} \cdot \underline{v} = 0$ Table 5.1 p. 111 (const ρ)

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r v_r)}_{0} + \underbrace{\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)}_{0} + \frac{\partial v_z}{\partial z} = 0$$

$\frac{\partial v_z}{\partial z} = 0$ ⟸ fully developed

Newtonian only $\quad$ p. 143, Table 6.5

$$\underline{\underline{\tau}} = 2\mu \underline{\underline{\Gamma}} = \mu\left[\underline{\nabla}\underline{v} + (\underline{\nabla}\underline{v})^{+}\right]$$

$$\tau_{rz} = \mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]$$

$$\tau_{rz,PL} = f\left(\frac{\partial v_z}{\partial r}, \frac{\partial v_r}{\partial z}\right)$$

$$\tau_{\theta z,Newt} = \mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\tau_{\theta z,PL} = f\left(\frac{\partial v_\theta}{\partial z}, \frac{\partial v_z}{\partial \theta}\right)$$

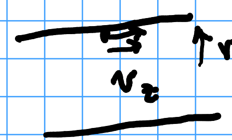
$\tau_{zz,Newt} = 2\mu \frac{\partial v_z}{\partial z}$

$\tau_{zz,PL} = f\left(\frac{\partial v_z}{\partial z}\right)$

$0 = -\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}\right)$ simplified!

$\tau = m s^n$

$\tau_{rz} = m\left(\underbrace{\frac{\partial v_z}{\partial r}}_{s}\right)^n$



$0 = -\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot m\left(\frac{\partial v_z}{\partial r}\right)^n\right)$

✓ no slip

$v_z(R) = 0$

$|v_z| < \infty$

$|\tau_{rz}| < \infty$

"Math":

$\frac{\partial P}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r m\left(\frac{\partial v_z}{\partial r}\right)^n\right) = C$

$P = P(z)$

$v_z = v_z(r)$

$\frac{\partial P}{\partial z} = \frac{\Delta P}{L}$

$\frac{1}{r}\frac{\partial}{\partial r}\left(r m\left(\frac{\partial v_z}{\partial r}\right)^n\right) = \frac{\Delta P}{L}$

$\int d\left(r m\left(\frac{\partial v_z}{\partial r}\right)^n\right) = \int \frac{\Delta P}{L} r \, dr$

$r m\left(\frac{\partial v_z}{\partial r}\right)^n = \frac{\Delta P}{L}\frac{r^2}{2} + C \longrightarrow$

$\tau_{rz} = \frac{\Delta P}{L}\frac{r}{2} + \frac{\overset{0}{C}}{r}$

$\tau_{rz} = \frac{\Delta P}{L}\frac{r}{2}$

$\left(\frac{\partial v_z}{\partial r}\right)^n = \frac{\Delta P}{mL}\frac{r}{2} + \cancel{\frac{C}{r}}$

$\frac{\partial v_z}{\partial r} = -\left(\frac{|\Delta P|}{mL}\frac{r}{2}\right)^{1/n} \leftarrow \Delta P < 0$

$$\int dv_z = -\int \left(\frac{|\Delta P|}{mL}\frac{r}{2}\right)^{1/n} dr$$

$$v_z = -\left(\frac{|\Delta P|}{2mL}\right)^{1/n} \int r^{1/n}\, dr$$

$$\frac{1}{n}+1 = \frac{1}{n}+\frac{n}{n} = \frac{n+1}{n}$$

$$= -\left(\frac{|\Delta P|}{2mL}\right)^{1/n} \frac{r^{1/n+1}}{\frac{1}{n}+1} + C_2$$

$$= -\left(\frac{|\Delta P|}{2mL}\right)^{1/n} \frac{r^{n+1/n}}{(n+1)/n} + C_2$$

$$v_z = \frac{-n}{n+1}\left(\frac{|\Delta P|}{2mL}\right)^{1/n} r^{n+1/n} + C_2$$

Eval B.C #2 (no slip)

$$v_z(r=R)=0 = \frac{-n}{n+1}\left(\frac{|\Delta P|}{2mL}\right)^{1/n} R^{\frac{n+1}{n}} + C_2$$

$$v_z = \frac{-n}{n+1}\left(\frac{|\Delta P|}{2mL}\right)^{1/n}\left(r^{\frac{n+1}{n}} - R^{n+1/n}\right)$$

non PL

$$= \frac{-n}{n+1}\left(\frac{R^{n+1}|\Delta P|}{2mL}\right)^{1/n}\left(\left(\frac{r}{R}\right)^{\frac{n+1}{n}} - 1\right)$$

$$\boxed{v_z = \frac{n}{n+1}\left(\frac{R^{n+1}|\Delta P|}{2mL}\right)^{1/n}\left(1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right)}$$

missing part from book

Eq. 7.5-8

$$f = ? = \frac{\tau_w}{1/2\,\rho u^2}$$ HW ?

$u \rightarrow$ Eq. 7.5-9