④ Derivation of f for power-law fluid

$$f = \frac{\tau_w}{\frac{1}{2}\rho u^2} \qquad \tau_w = \tau(r=R) = m\left(\frac{\partial v_z}{\partial r}\right)^n\Big|_{r=R}$$

$$\tau_w = m\left\{\frac{\partial}{\partial r}\left(\frac{3n+1}{n+1}\right)u\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]\right\}^n\Big|_{r=R}$$

$$= m\left\{\frac{3n+1}{n+1}u\left[-\frac{n+1}{n}\frac{1}{R}\left(\frac{r}{R}\right)^{\frac{n+1}{n}-1}\right]\right\}^n\Big|_{r=R}$$

$$= m\left\{-\frac{3n+1}{\cancel{n+1}}\frac{u}{R}\left(\frac{\cancel{n+1}}{n}\right)\left(\frac{r}{R}\right)^{1/n}\right\}^n\Big|_{r=R}$$

$$= -m\left\{\frac{3n+1}{n}\frac{u}{R}\left(\frac{r}{R}\right)^{1/n}\right\}^n\Big|_{r=R}$$

$$= -m\left(\frac{3n+1}{n}\frac{u}{R}\right)^n \underbrace{\frac{r}{R}}_{1}\Big|_{r=R}$$

$$\tau_w = -m\left(\frac{3n+1}{n}\frac{u}{R}\right)^n$$

$$f = \frac{2|\tau_w|}{\rho u^2} = 2m\left(\frac{3n+1}{n}\frac{u}{R}\right)^n\frac{1}{\rho u^2}$$

$$f = 2m\left(\frac{(3n+1)}{n}\frac{2}{D}\right)^n u^n\frac{1}{\rho u^2}$$

$$\boxed{f = 2\left(\frac{2(3n+1)}{n}\right)^n\frac{m}{\rho u^{2-n}D^n}}$$

$$Re_{PL} = \frac{16}{f} = \frac{16}{2\left(\frac{2(3n+1)}{n}\right)^n}\frac{\rho u^{2-n}D^n}{m}$$

$$\boxed{Re_{PL} = \frac{8\rho u^{2-n}D^n}{\left[\frac{2(3n+1)}{n}\right]^n m}}$$

(*) Picking reference viscosity to be at $r=R$ (at the wall)

$$\tau = \mu s \Rightarrow \tau = m s^n = (m s^{n-1}) s$$

$$\mu_{eff} = m s^{n-1}$$

$$= m \left(\frac{\partial v_z}{\partial r}\right)^{n-1}$$

use derivative on p. 31-5

$$\mu_{eff} = -m \left[\frac{3n+1}{n} \frac{u}{R} \left(\frac{r}{R}\right)^{1/n} \right]^{n-1}$$

$r = R$

$$\mu_{eff,wall} = -m \left[\frac{3n+1}{n} \frac{u}{R} \right]^{n-1}$$

$$\mu_w = -m \left[\frac{2(3n+1)}{n} \frac{u}{D} \right]^{n-1}$$

now, let $Re_w = \dfrac{\rho u D}{|\mu_w|}$

$$= \rho u D \left[\frac{1}{m} \left\{ \frac{2(3n+1)}{n} \frac{u}{D} \right\}^{-n+1} \right]$$

$$= \frac{\rho u D}{m} \left[\frac{2(3n+1)}{n} \right]^{-n+1} \frac{u^{-n+1}}{D^{-n+1}} = \left[\frac{2(3n+1)}{n}\right]^{-n+1} \frac{\rho u^{2-n} D^n}{m}$$

$$\boxed{Re_w = \left[\frac{n}{2(3n+1)} \right]^{n-1} \frac{\rho u^{2-n} D^n}{m}}$$

$$\frac{Re_{PL}}{Re_w} = \frac{\left[\frac{n}{2(3n+1)}\right]^n \frac{8 \rho u^{2-n} D}{m}}{\left[\frac{n}{2(3n+1)}\right]^{n-1} \frac{\rho u^{2-n} D}{m}} = \frac{8n}{2(3n+1)} = \frac{4n}{3n+1}$$

$$\boxed{Re_{PL} = \frac{4n}{3n+1} Re_w}$$