Lecture 17 - Supplement: Derivation of Navier Stokes

* Navier-stokes is a force balance written in a slightly different way: as a momentum balance.

> See also Deen 6.4, 6.6

$$\sum_i \underline{F}_i = m\underline{a} \qquad \underline{a} = \frac{d\underline{v}}{dt}$$

if mass is constant then we can pull it into $\frac{d}{dt}$

$$\sum_i \underline{F}_i = m\frac{d\underline{v}}{dt} = \frac{d}{dt}(m\underline{v})$$

momentum, $\underline{p}$

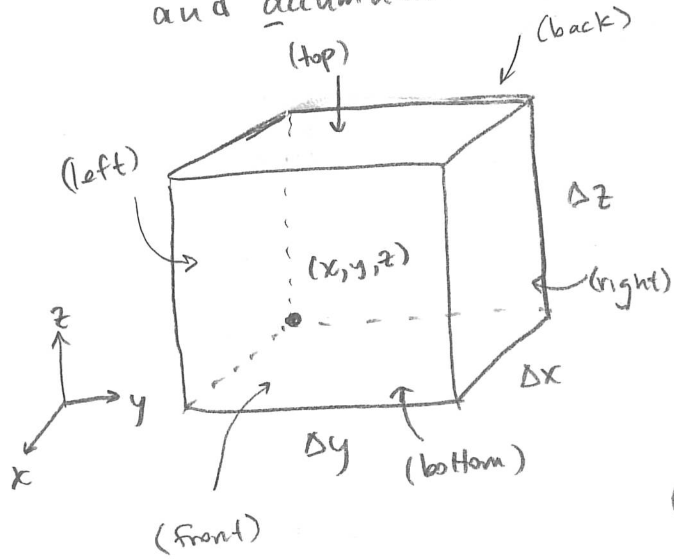
$$\sum_i \underline{F}_i = \frac{d\underline{p}}{dt}$$

generation terms

accumulation term

what generates momentum? Forces!

* A balance equation will have input, output, generation and accumulation terms: IOGA.



accum = in - out + generation

$$\underbrace{\frac{d}{dt}(\underline{p})}_{\text{control volume}} = \underbrace{\dot{\underline{p}}_{in} - \dot{\underline{p}}_{out}}_{\text{momentum flow}} + \underbrace{\sum_i \underline{F}_i}_{\text{sum of forces on C.V.}}$$

(*) Notice this is a vector equation!

A. Accumulation term.

$$\frac{\partial}{\partial t}(\underline{P}) = \frac{\partial}{\partial t}(m\underline{v}) = \frac{\partial}{\partial t}(\rho \underbrace{\Delta x \Delta y \Delta z}_{\text{box volume.}} \underline{v})$$

partial derivative b/c we will have other space derivatives

B. In-Out terms.

what is the momentum flow: $\dot{\underline{P}}_{in} - \dot{\underline{P}}_{out}$?

- recall mass flow: $\dot{m} = \rho Q = \rho u A = \underbrace{\rho \underline{v}}_{\text{mass flux}} \cdot (-\underline{n} A)$ mass flux $= \frac{\text{mass}}{\text{area time}}$

negative b/c we want "in" to be positive

- by analogy we can write for momentum: $\dot{\underline{p}} = \dot{m}\underline{v} = \rho Q \underline{v} = \rho u \underline{v} A$

$$= \underbrace{\rho \underline{v}\underline{v}}_{\text{momentum flux}} \cdot (-\underline{n} A)$$

momentum flux $= \frac{\text{momentum}}{\text{area time}}$

dyadic

$$\rho \underline{v}\underline{v} = \rho \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix}$$

* Now we need to evaluate the momentum flow on all 6 sides of our control volume:

(back) $\quad -(\rho \underline{v}\underline{v})|_x \cdot (-\underline{e}_x)\Delta y \Delta z = (\rho \underline{v} v_x)|_x \Delta y \Delta z$

(front) $\quad -(\rho \underline{v}\underline{v})|_{x+\Delta x} \cdot (\underline{e}_x)\Delta y \Delta z = -(\rho \underline{v} v_x)|_{x+\Delta x} \Delta y \Delta z$

(left) $-(\rho \underline{v}\underline{v})|_y \cdot (-\underline{e}_y)\Delta x \Delta z = (\rho \underline{v} v_y)|_y \Delta x \Delta z$

(right) $-(\rho \underline{v}\underline{v})|_{y+\Delta y} \cdot (\underline{e}_y)\Delta x \Delta z = -(\rho \underline{v} v_y)|_{y+\Delta y} \Delta x \Delta z$

(bottom) $-(\rho \underline{v}\underline{v})|_z \cdot (-\underline{e}_z)\Delta x \Delta y = (\rho \underline{v} v_z)|_z \Delta x \Delta y$

(top) $-(\rho \underline{v}\underline{v})|_{z+\Delta z} \cdot (\underline{e}_z)\Delta x \Delta y = -(\rho \underline{v} v_z)|_{z+\Delta z} \Delta x \Delta y$

* all together:

$$\dot{\underline{P}}_{in} - \dot{\underline{P}}_{out} = \left[(\rho \underline{v} v_x)|_x - (\rho \underline{v} v_x)|_{x+\Delta x}\right]\Delta y \Delta z$$
$$+ \left[(\rho \underline{v} v_y)|_y - (\rho \underline{v} v_y)|_{y+\Delta y}\right]\Delta x \Delta z$$
$$+ \left[(\rho \underline{v} v_z)|_z - (\rho \underline{v} v_z)|_{z+\Delta z}\right]\Delta x \Delta y$$

C. Generation

* Now we need to look at the sum of the forces on our control volume:

$$\sum_i \underline{F} = \underbrace{\underline{F}_s}_{\text{surface forces}} + \underbrace{\underline{F}_b}_{\text{body forces}}$$

$\underline{F}_b = m\underline{g}$ ← gravitational forces

$$\underline{F}_s = A\underline{n} \cdot \underline{\underline{\sigma}} = A(\underbrace{\underline{n}P}_{\text{pressure}} + \underbrace{\underline{n} \cdot \underline{\underline{\tau}}}_{\text{viscous stresses}})$$

Body Forces: $\underline{F}_b = m\underline{g} = \rho \Delta x \Delta y \Delta z \underline{g}$

Surface forces: we have 6 surfaces!

(back) $-\underline{e}_x \cdot \underline{\underline{\sigma}}|_x \Delta y \Delta z$

(front) $\underline{e}_x \cdot \underline{\underline{\sigma}}|_{x+\Delta x} \Delta y \Delta z$

(left) $-\underline{e}_y \cdot \underline{\underline{\sigma}}|_y \Delta x \Delta z$

(right) $\underline{e}_y \cdot \underline{\underline{\sigma}}|_{y+\Delta y} \Delta x \Delta z$

(bottom) $-\underline{e}_z \cdot \underline{\underline{\sigma}}|_z \Delta x \Delta y$

(top) $\underline{e}_z \cdot \underline{\underline{\sigma}}|_{z+\Delta z} \Delta x \Delta y$

* combining the forces gives:

$$\sum_i F_i = \rho \Delta x \Delta y \Delta z \underline{g} + \underline{e}_x \cdot \left(\underline{\underline{\sigma}}|_{x+\Delta x} - \underline{\underline{\sigma}}|_x\right) \Delta y \Delta z + \underline{e}_y \left(\underline{\underline{\sigma}}|_{y+\Delta y} - \underline{\underline{\sigma}}|_y\right) \Delta x \Delta z + \underline{e}_z \left(\underline{\underline{\sigma}}|_{z+\Delta z} - \underline{\underline{\sigma}}|_z\right) \Delta x \Delta y$$

D. Cauchy Momentum Equation

$$\frac{\partial}{\partial t}(\underline{P}) = \dot{\underline{P}}_{in} - \dot{\underline{P}}_{out} + \sum_i F_i$$

combining all of our previous pieces together

we get:

$$\frac{\partial}{\partial t}(\rho\,\Delta x\,\Delta y\,\Delta z\,\underline{v}) = -\left[(\rho\underline{v}v_x)|_{x+\Delta x} - (\rho\underline{v}v_x)|_x\right]\Delta y\,\Delta z$$
$$-\left[(\rho\underline{v}v_y)|_{y+\Delta y} - (\rho\underline{v}v_y)|_y\right]\Delta x\,\Delta z$$
$$-\left[(\rho\underline{v}v_z)|_{z+\Delta z} - (\rho\underline{v}v_z)|_z\right]\Delta x\,\Delta y$$
$$+\,\underline{e}_x\cdot(\underline{\underline{\sigma}}|_{x+\Delta x} - \underline{\underline{\sigma}}|_x)\,\Delta y\,\Delta z$$
$$+\,\underline{e}_y\cdot(\underline{\underline{\sigma}}|_{y+\Delta y} - \underline{\underline{\sigma}}|_y)\,\Delta x\,\Delta z$$
$$+\,\underline{e}_z\cdot(\underline{\underline{\sigma}}|_{z+\Delta z} - \underline{\underline{\sigma}}|_z)\,\Delta x\,\Delta y$$
$$+\,\rho\,\Delta x\,\Delta y\,\Delta z\,\underline{g}$$

- divide by $\Delta x\,\Delta y\,\Delta z$
- take the limit as $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$

$$\frac{(\rho\underline{v}v_x)|_{x+\Delta x} - (\rho\underline{v}v_x)|_x}{\Delta x} \to \frac{\partial}{\partial x}(\rho\underline{v}v_x)$$

and so on...

$$\frac{\partial}{\partial t}(\rho\underline{v}) = -\frac{\partial}{\partial x}(\rho\underline{v}v_x) - \frac{\partial}{\partial y}(\rho\underline{v}v_y) - \frac{\partial}{\partial z}(\rho\underline{v}v_z)$$
$$+\,\underline{e}_x\cdot\frac{\partial\underline{\underline{\sigma}}}{\partial x} + \underline{e}_y\cdot\frac{\partial\underline{\underline{\sigma}}}{\partial y} + \underline{e}_z\cdot\frac{\partial\underline{\underline{\sigma}}}{\partial z} + \rho\underline{g}$$

Can write these guys more compactly: $\underline{\nabla}\cdot\underline{\underline{f}} = \underline{e}_x\cdot\frac{\partial}{\partial x}\underline{\underline{f}} + \underline{e}_y\cdot\frac{\partial}{\partial y}\underline{\underline{f}} + \underline{e}_z\cdot\frac{\partial}{\partial z}\underline{\underline{f}}$

$$\frac{\partial}{\partial t}(\rho\underline{v}) = -\underline{\nabla}\cdot(\rho\underline{v}\underline{v}) + \underline{\nabla}\cdot\underline{\underline{\sigma}} + \rho\underline{g}$$ (phew!)

$$\underbrace{\frac{\partial}{\partial t}(\rho\underline{v}) + \underline{\nabla}\cdot(\rho\underline{v}\underline{v})} = \underline{\nabla}\cdot\underline{\underline{\sigma}} + \rho\underline{g}$$

this term will simplify

$$\frac{\partial}{\partial t}(\rho \underline{v}) = \rho \frac{\partial \underline{v}}{\partial t} + \underline{v} \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot (\rho \underline{v}\underline{v}) = \rho \underline{v} \cdot \underline{\nabla} \underline{v} + \underline{v}\, \underline{\nabla} \cdot (\rho \underline{v})$$

This is the product rule in vector calculus

(*) See appendix

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v}\underline{v}) = \rho \frac{\partial \underline{v}}{\partial t} + \underline{v} \frac{\partial \rho}{\partial t} + \rho \underline{v} \cdot \underline{\nabla} \underline{v} + \underline{v}\, \underline{\nabla} \cdot (\rho \underline{v})$$

$$= \underline{v} \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v})\right]}_{\text{continuity equation}} + \rho \underbrace{\left[\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v}\right]}_{\text{material derivative}}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \qquad \rho \frac{D\underline{v}}{Dt}$$

$$\boxed{\rho \frac{D\underline{v}}{Dt} = \underline{\nabla} \cdot \underline{\underline{\sigma}} + \rho \underline{g}}$$

Cauchy's Momentum Equation, "Newton's 2nd law for fluids"

E. Navier-Stokes Equation

* 2 assumptions :

(1) Newtonian Fluid (constant μ)

(2) Incompressible fluid (constant ρ)

* Both of these will let us simplify $\underline{\underline{\sigma}}$:

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} = \underline{\nabla} \cdot (-P \underline{\underline{\delta}} + \underline{\underline{\tau}})$$

Newtonian Fluid

$$\underline{\underline{\tau}} = 2\mu \underline{\underline{\Gamma}} = \mu\left[(\nabla \underline{v}) + (\nabla \underline{v})^T\right]$$

also assumes incompressible.

$$\underline{\nabla}\cdot\underline{\underline{\sigma}} = \nabla\cdot\left\{-P\underline{\underline{\delta}} + \mu\left[(\underline{\nabla}\underline{v}) + (\underline{\nabla}\underline{v})^T\right]\right\}$$

$$= -\underline{\nabla}P + \mu\,\underline{\nabla}\cdot\left[(\underline{\nabla}\underline{v}) + (\underline{\nabla}\underline{v})^T\right]$$

(*) see appendix for proof of vector property.

$$= -\underline{\nabla}P + \mu\,\underbrace{\underline{\nabla}\cdot(\underline{\nabla}\underline{v})}_{\underline{\nabla}\cdot\underline{\nabla}=\nabla^2} + \mu\underline{\nabla}(\underline{\nabla}\cdot\underline{v})$$

Incompressible fluid

$\underline{\nabla}\cdot\underline{v} = 0$

$$= -\underline{\nabla}P + \mu\nabla^2\underline{v}$$

Finally,

$$\boxed{\rho\frac{D\underline{v}}{Dt} = -\underline{\nabla}P + \mu\nabla^2\underline{v} + \rho\underline{g}}$$

← The Navier-Stokes Equation!

- Newton's 2nd law
  + Newtonian
  + Incompressible.

✓ in Cartesian coordinates (p. 147 in §6.6)

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$$\rho\left[\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right] = -\frac{\partial P}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

$$\rho\left[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial P}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

F. Appendix : Proof of vector Identities

Einstein Index Notation

$$\nabla \cdot (\rho \underline{v}\,\underline{v}) = \partial_i (\rho v_i v_j)$$

$$= v_j \partial_i (\rho v_i) + \rho v_i \partial_i (v_j) \quad \text{product rule}$$

$$= \underline{v}\, \nabla \cdot (\rho \underline{v}) + \rho \underline{v} \cdot \underline{\nabla}\, \underline{v}$$

$$\underline{\nabla} \cdot [(\underline{\nabla}\underline{v}) + (\underline{\nabla}\underline{v})^T] = \partial_i [\partial_i v_j + \partial_j v_i]$$

$$= \partial_i \partial_i v_j + \partial_i \partial_j v_i$$

$$= \partial_i \partial_i v_j + \partial_j \partial_i v_i$$

$$= \nabla \cdot \nabla (\underline{v}) + \nabla (\nabla \cdot \underline{v})$$

$$= \nabla^2 \underline{v} + \underline{\nabla} (\underline{\nabla} \cdot \underline{v})$$