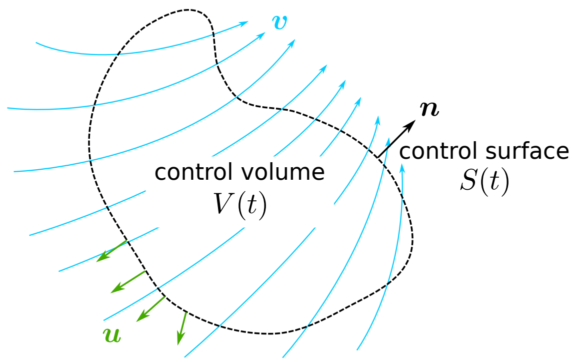
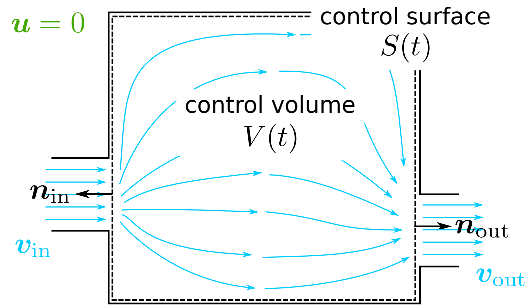
General Control Volume



Discrete Inlet/Outlet Control Volume



I. Mass Balance

General Mass Balance

$$\frac{d}{dt}\int_{V(t)} \rho \, dV = -\int_{S(t)} \rho(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{n}\, dS$$

Engineering Mass Balance

$$\frac{dm}{dt} = \sum_{i}^{\text{inlets}} w_i - \sum_{i}^{\text{outlets}} w_i$$

Assumptions

- Discrete inlets & outlets
- Uniform density at inlets & outlets

Comments

- m is the mass of the control volume.

II. Momentum Balance

General Momentum Balance

$$\frac{d}{dt}\int_{V(t)} \rho\boldsymbol{v}\, dV = -\int_{S(t)} \rho\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{n}\, dS + m\boldsymbol{g} - \int_{S(t)} \boldsymbol{n}P\, dS + \int_{S(t)} \boldsymbol{n}\cdot\boldsymbol{\tau}\, dS$$

Engineering Momentum Balance

$$\frac{d(m\boldsymbol{v})}{dt} = \sum_{i}^{\text{inlets}} a_i w_i \boldsymbol{v}_i - \sum_{i}^{\text{outlets}} a_i w_i \boldsymbol{v}_i + \sum_i \boldsymbol{F}_i$$

Assumptions

- Discrete inlets & outlets
- Unidirectional flow at inlets & outlets
- Small viscious stress at inlets & outlets
- Uniform density at inlets & outlets
- Fixed control volume ($u = 0$)

Comments

- This is a *vector* equation.
- $a_i = \begin{cases} 4/3 \text{ for laminar flow} \\ 50/49 \approx 1 \text{ for turbulent flow} \\ 1 \text{ for plug/uniform flow} \end{cases}$
- v_i are average velocities.
- $\boldsymbol{F}_i$ are forces on the CV.
- Viscous stresses can be safely neglected for pipe entrances and exits, but not when the control surface crosses (i) sudden contractions or expansions, (ii) fans/pumps/turbines, (iii) turbulent wakes.

III. Mechanical Energy Balance

General Mechanical Energy Balance

$$\frac{d}{dt}\int_{V(t)} \rho \left[\frac{v^2}{2} + gh\right] dV + \int_{S(t)} \rho \left[\frac{v^2}{2} + gh\right] (\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{n}\, dS = \\ - \int_{S(t)} P(\boldsymbol{n} \cdot \boldsymbol{v})\, dS + \int_{S(t)} \boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \boldsymbol{v}\, dS + \int_{S(t)} P(\nabla \cdot \boldsymbol{v})\, dS - \int_{S(t)} \boldsymbol{\tau} : \nabla \boldsymbol{v}\, dS.$$

Engineering Bernoulli Equation

$$\left(\frac{bv^2}{2} + \frac{P}{\rho} + gh\right)_{\text{outlet}} - \left(\frac{bv^2}{2} + \frac{P}{\rho} + gh\right)_{\text{inlet}} = \frac{1}{w}\left(W_m - E_v\right)$$

Assumptions

- *Single* inlet & outlet
- Unidirectional flow at inlets & outlets
- Small viscious stress at inlets & outlets
- Constant density
- Fixed control volume ($u = 0$)
- *Steady*

Comments

- W_m is the *shaft work* by devices adding or removing energy from the CV.
- E_v are losses due to viscous friction
- In pipes, $E_v = Q\,|\Delta \mathcal{P}| = w\dfrac{2U^2Lf}{D}$
- $b = \begin{cases} 2 \text{ for laminar flow} \\ 1.06 \approx 1 \text{ for turbulent flow} \\ 1 \text{ for plug/uniform flow} \end{cases}$
- For a streamline in inviscid flow $b = 1$ and $W_m = E_v = 0$. This gives Bernoulli's equation.