Solution

Navier Stokes in cylindrical coordinates in the r and θ directions

$*u = u(r)$

$$\rho\left(\cancel{\frac{\partial u_r}{\partial t}} + u_r\frac{\partial u_r}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}} - \frac{u_\theta^2}{r} + \cancel{u_z\frac{\partial u_r}{\partial z}}\right)$$

steady independent two-dimensional

$$= -\frac{\partial P}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) - \frac{u_r}{r^2} + \cancel{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 u_r}{\partial z^2}}\right]$$

independent independent two-dimensional

$$\rho\left(\cancel{\frac{\partial u_\theta}{\partial t}} + u_r\frac{\partial u_\theta}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{u_r u_\theta}{r} + \cancel{u_z\frac{\partial u_\theta}{\partial z}}\right)$$

steady Independent two-dimensional

$$= \frac{-1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2} + \cancel{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}} + \cancel{\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}} + \cancel{\frac{\partial^2 u_\theta}{\partial z^2}}\right]$$

independent Independent two-dimensional

Evaluate derivatives

$u_r = C/r \qquad u_\theta = K/r$

$$\frac{\partial u_r}{\partial r} = \frac{-c}{r^2} \qquad \frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) = \frac{\partial}{\partial r}\left(r\frac{-c}{r^2}\right) = \frac{\partial}{\partial r}\left(\frac{-c}{r}\right) = \frac{c}{r^2}$$

$$\frac{\partial u_\theta}{\partial r} = \frac{-K}{r^2} \qquad \frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) = K/r^2$$

Substitute and solve

$$\rho\left[\left(\frac{c}{r}\right)\left(\frac{-c}{r^2}\right) - \left(\frac{K^2}{r^2}\right)\left(\frac{1}{r}\right)\right] = \frac{-\partial P}{\partial r} + \mu\left[\left(\frac{1}{r}\right)\left(\frac{c}{r^2}\right) - \left(\frac{c}{r}\right)\left(\frac{1}{r^2}\right)\right]$$

$$\rho\left(\frac{-c^2}{r^3} - \frac{K^2}{r^3}\right) = \frac{-\partial P}{\partial r}$$ Separate and integrate

$$\int \rho\,(c^2+K^2)\frac{1}{r^3}\,dr = \int dP \Rightarrow P = \frac{-1}{2}(c^2+K^2)\frac{1}{r^2} + C_1$$

$$\rho\left[\left(\frac{c}{r}\right)\left(\frac{-K}{r^2}\right) + \left(\frac{c}{r}\right)\left(\frac{K}{r}\right)\left(\frac{1}{r}\right)\right] = \frac{-1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\left(\frac{1}{r}\right)\left(\frac{K}{r^2}\right) - \left(\frac{K}{r}\right)\left(\frac{1}{r^2}\right)\right]$$

$$0 = \frac{-1}{r}\frac{\partial P}{\partial \theta}$$

$$\int 0\,d\theta = \int dP \Rightarrow P = C_2$$

Combine $\boxed{P = \frac{-1}{2}\frac{c^2+K^2}{r^2} + C_3}$