Solutions Manual for

Fluid Mechanics: Fundamentals and Applications

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Chapter 9

DIFFERENTIAL ANALYSIS OF FLUID FLOW

General and Mathematical Background Problems

9-1C

Solution We are to express the divergence theorem in words.

Analysis For vector $\vec{G}$, **the volume integral of the divergence of $\vec{G}$ over volume *V* is equal to the surface integral of the normal component of $\vec{G}$ taken over the surface *A* that encloses the volume**.

Discussion The divergence theorem is also called *Gauss's theorem*.

9-2C

Solution We are to explain the fundamental differences between a flow domain and a control volume.

Analysis **A *control volume* is used in an integral, control volume solution**. It is a volume over which all mass flow rates, forces, etc. are specified over the entire control surface of the control volume. In a control volume analysis we do not know or care about details *inside* the control volume. Rather, we solve for gross features of the flow such as net force acting on a body. **A *flow domain,* on the other hand, is also a volume, but is used in a differential analysis**. Differential equations of motion are solved everywhere inside the flow domain, and we *are* interested in all the details inside the flow domain.

Discussion Note that we also need to specify what is happening at the boundaries of a flow domain – these are called *boundary conditions*.

9-3C

Solution We are to explain what we mean by coupled differential equations.

Analysis A set of coupled differential equations simply means that **the equations are dependent on each other and must be solved together rather than separately**. For example, the equations of motion for fluid flow involve velocity variables in both the conservation of mass equation and the momentum equation. To solve for these variables, we must solve the coupled set of differential equations together.

Discussion In some very simple fluid flow problems, the equations become uncoupled, and are easier to solve.

9-4C

Solution We are to discuss the number of unknowns and the equations needed to solve for those unknowns for a three-dimensional, unsteady, incompressible flow field.

Analysis There are **four unknowns** (velocity components *u*, *v*, *w*, and pressure *P*) and thus we need to solve **four equations**:

- one from conservation of mass which is a scalar equation
- three from Newton's second law which is a vector equation

Discussion These equations are also coupled in general.

9-5C

Solution We are to discuss the number of unknowns and the equations needed to solve for those unknowns for a two-dimensional, unsteady, compressible flow field with significant variations in both temperature and density.

Analysis There are **five unknowns** (velocity components u and v, and ρ, T, and P) and thus we need to solve **five equations**:

- one from conservation of mass which is a scalar equation
- two from Newton's second law which is a vector equation
- one from the energy equation which is a scalar equation
- one from an equation of state (e.g., ideal gas law) which is a scalar equation

Discussion These equations are also coupled in general.

9-6C

Solution We are to discuss the number of unknowns and the equations needed to solve for those unknowns for a two-dimensional, unsteady, incompressible flow field.

Analysis There are **three unknowns** (velocity components u, and v, and pressure P) and thus we need to solve **three equations**:

- one from conservation of mass which is a scalar equation
- two from Newton's second law which is a vector equation

Discussion These equations are also coupled in general.

9-7

Solution We are to transform a position from Cartesian to cylindrical coordinates.

Analysis We use the coordinate transformations provided in this chapter,

$$r = \sqrt{x^2 + y^2} = \sqrt{(2\text{ m})^2 + (4\text{ m})^2} = 4.47214\text{ m} \tag{1}$$

and

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{4\text{ m}}{2\text{ m}}\right) = 63.43495° = 1.10715\text{ radians} \tag{2}$$

Coordinate z remains unchanged. Thus, to three significant digits,

Position in cylindrical coordinates: $$\vec{x} = (r,\theta,z) = (\mathbf{4.47\text{ m}, 1.11\text{radians}, -1\text{m}}) \tag{3}$$

Discussion Notice that the units of θ are radians – a dimensionless unit.

9-8

Solution We are to transform a position from cylindrical to Cartesian coordinates.

Analysis We use the coordinate transformations provided in this chapter,

$$x = r\cos\theta = (5\text{ m})\cos(\pi/3\text{ radians}) = (5\text{ m})\cos(60^{\circ}) = 2.5\text{ m} \qquad (1)$$

and

$$y = r\sin\theta = (5\text{ m})\sin(\pi/3\text{ radians}) = (5\text{ m})\sin(60^{\circ}) = 4.33013\text{ m} \qquad (2)$$

Coordinate z remains unchanged. Thus, to three significant digits,

Position in cylindrical coordinates: $$\vec{x} = (x,y,z) = \mathbf{(2.50\text{ m}, 4.33\text{ m}, 1.27\text{ m})} \qquad (3)$$

Discussion You can verify your answer by using the reverse equations, as in the previous problem.

9-9

Solution We are to calculate a truncated Taylor series expansion for a given function and compare our result with the exact value.

Analysis The algebra here is simple since $d(e^x)/dx = e^x$. The Taylor series expansion is

Taylor series expansion: $$f(x_0 + dx) = e^{x_0} + e^{x_0}dx + \frac{1}{2}e^{x_0}dx^2 + \frac{1}{3\times 2}e^{x_0}dx^3 + ... \qquad (1)$$

We plug $x_0 = 0$ and $dx = -0.1$ into Eq. 1,

Truncated Taylor series expansion:

$$f(-0.1) \approx 1 + 1\times(-0.1) + \frac{1}{2}\times 1\times(-0.1)^2 + \frac{1}{6}\times 1\times(-0.1)^3 = 0.9048333... \qquad (2)$$

We compare Eq. 2 with the exact value,

Exact value: $$f(-0.1) = e^{-0.1} = 0.904837418... \qquad (3)$$

Comparing Eqs. 2 and 3 we see that **our approximation is good to four or five significant digits**.

Discussion The smaller the value of dx, the better the approximation. You can easily convince yourself of this by trying $dx = 0.01$ instead.

9-10

Solution We are to calculate the divergence of a given vector.

Analysis The divergence of $\vec{G}$ is the dot product of the del operator $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ with $\vec{G}$, which gives

Divergence of $\vec{G}$: $$\vec{\nabla}\cdot\vec{G} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(2xz\vec{i} - \frac{1}{2}x^2\vec{j} - z^2\vec{k}\right) = 2z + 0 - 2z = \mathbf{0}$$

It turns out that for this special case, the divergence of $\vec{G}$ is zero.

Discussion If $\vec{G}$ were a velocity vector, this would mean that the flow field is incompressible.

9-11

Solution We are to expand the given equation in Cartesian coordinates and verify it.

Analysis In Cartesian coordinates the del operator is $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ and we let $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ and $\vec{G} = G_x\vec{i} + G_y\vec{j} + G_z\vec{k}$. The left hand side of the equation is thus

$$\begin{aligned}
\vec{\nabla}\cdot\left(\vec{F}\vec{G}\right) &= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} F_xG_x & F_xG_y & F_xG_z \\ F_yG_x & F_yG_y & F_yG_z \\ F_zG_x & F_zG_y & F_zG_z \end{bmatrix} \\
&= \left[\frac{\partial}{\partial x}\left(F_xG_x\right) + \frac{\partial}{\partial y}\left(F_yG_x\right) + \frac{\partial}{\partial z}\left(F_zG_x\right)\right]\vec{i} \\
&+ \left[\frac{\partial}{\partial x}\left(F_xG_y\right) + \frac{\partial}{\partial y}\left(F_yG_y\right) + \frac{\partial}{\partial z}\left(F_zG_y\right)\right]\vec{j} \\
&+ \left[\frac{\partial}{\partial x}\left(F_xG_z\right) + \frac{\partial}{\partial y}\left(F_yG_z\right) + \frac{\partial}{\partial z}\left(F_zG_z\right)\right]\vec{k}
\end{aligned} \tag{1}$$

We use the product rule on each term in Eq. 1 and rearrange to get

Left hand side:

$$\begin{aligned}
\vec{\nabla}\cdot\left(\vec{F}\vec{G}\right) &= \left[G_x\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) + \left(F_x\frac{\partial}{\partial x} + F_y\frac{\partial}{\partial y} + F_z\frac{\partial}{\partial z}\right)G_x\right]\vec{i} \\
&+ \left[G_y\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) + \left(F_x\frac{\partial}{\partial x} + F_y\frac{\partial}{\partial y} + F_z\frac{\partial}{\partial z}\right)G_y\right]\vec{j} \\
&+ \left[G_z\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) + \left(F_x\frac{\partial}{\partial x} + F_y\frac{\partial}{\partial y} + F_z\frac{\partial}{\partial z}\right)G_z\right]\vec{k}
\end{aligned} \tag{2}$$

We recognize that $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \vec{\nabla}\cdot\vec{F}$ and $F_x\frac{\partial}{\partial x} + F_y\frac{\partial}{\partial y} + F_z\frac{\partial}{\partial z} = \vec{F}\cdot\vec{\nabla}$. Eq. 2 then becomes

Left hand side:

$$\begin{aligned}
\vec{\nabla}\cdot\left(\vec{F}\vec{G}\right) &= \left[G_x\left(\vec{\nabla}\cdot\vec{F}\right) + \left(\vec{F}\cdot\vec{\nabla}\right)G_x\right]\vec{i} + \left[G_y\left(\vec{\nabla}\cdot\vec{F}\right) + \left(\vec{F}\cdot\vec{\nabla}\right)G_y\right]\vec{j} \\
&+ \left[G_z\left(\vec{\nabla}\cdot\vec{F}\right) + \left(\vec{F}\cdot\vec{\nabla}\right)G_z\right]\vec{k}
\end{aligned} \tag{3}$$

After rearrangement, Eq. 3 becomes

Left hand side:

$$\vec{\nabla}\cdot\left(\vec{F}\vec{G}\right) = \left(G_x\vec{i} + G_y\vec{j} + G_z\vec{k}\right)\left(\vec{\nabla}\cdot\vec{F}\right) + \left(\vec{F}\cdot\vec{\nabla}\right)\left(G_x\vec{i} + G_y\vec{j} + G_z\vec{k}\right) \tag{4}$$

Finally, recognizing vector $\vec{G}$ twice in Eq. 4, we see that the left hand side of the given equation is identical to the right hand side, and **the given equation is verified**.

Discussion It may seem surprising, but $\vec{F}\vec{G} \neq \vec{G}\vec{F}$.

9-12

Solution We are to prove the equation.

Analysis We let $\vec{F} = \rho\vec{V}$ and $\vec{G} = \vec{V}$. Using Eq. 1 of the previous problem, we have

$$\vec{\nabla}\cdot\left(\rho\vec{V}\vec{V}\right)=\vec{V}\vec{\nabla}\cdot\left(\rho\vec{V}\right)+\left(\rho\vec{V}\cdot\vec{\nabla}\right)\vec{V} \qquad (1)$$

However, since the density is not operated on in the second term of Eq. 1, it can be brought outside of the parenthesis, even though it is not a constant in general. Equation 1 can thus be written as

$$\boxed{\vec{\nabla}\cdot\left(\rho\vec{V}\vec{V}\right)=\vec{V}\vec{\nabla}\cdot\left(\rho\vec{V}\right)+\rho\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}} \qquad (2)$$

Discussion Equation 2 was used in this chapter in the derivation of the alternative form of Cauchy's equation.

9-13

Solution We are to transform cylindrical velocity components to Cartesian velocity components.

Analysis We apply trigonometry, recognizing that the angle between u and u_r is θ, and the angle between v and u_θ is also θ,

x component of velocity: $$\boxed{u = u_r\cos\theta - u_\theta\sin\theta} \qquad (1)$$

Similarly,

y component of velocity: $$\boxed{v = u_r\sin\theta + u_\theta\cos\theta} \qquad (2)$$

The transformation of the z component is trivial,

z component of velocity: $$\boxed{w = u_z} \qquad (3)$$

Discussion These transformations come in handy.

9-14

Solution We are to transform Cartesian velocity components to cylindrical velocity components.

Analysis We apply trigonometry, recognizing that the angle between u and u_r is θ, and the angle between v and u_θ is also θ,

u_r component of velocity: $$\boxed{u_r = u\cos\theta + v\sin\theta} \qquad (1)$$

Similarly,

u_θ component of velocity:: $$\boxed{u_\theta = -u\sin\theta + v\cos\theta} \qquad (2)$$

The transformation of the z component is trivial,

z component of velocity: $$\boxed{u_z = w} \qquad (3)$$

Discussion You can also obtain Eqs. 1 and 2 by solving Eqs. 1 and 2 of the previous problem simultaneously.

9-15

Solution We are to transform a given set of Cartesian coordinates and velocity components into cylindrical coordinates and velocity components.

Analysis First we apply the coordinate transformations given in this chapter,

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(0.40\text{ m}\right)^2 + \left(0.20\text{ m}\right)^2} = 0.4472\text{ m} \qquad (1)$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{0.20\text{ m}}{0.40\text{ m}}\right) = 26.565^\circ = 0.4636\text{ radians} \qquad (2)$$

Next we apply the results of the previous problem,

$$u_r = u\cos\theta + v\sin\theta = 10.3\frac{\text{m}}{\text{s}}\times\frac{0.40\text{ m}}{0.4472\text{ m}} - 5.6\frac{\text{m}}{\text{s}}\times\frac{0.20\text{ m}}{0.4472\text{ m}} = 6.708\frac{\text{m}}{\text{s}} \qquad (3)$$

$$u_\theta = -u\sin\theta + v\cos\theta = -10.3\frac{\text{m}}{\text{s}}\times\frac{0.20\text{ m}}{0.4472\text{ m}} - 5.6\frac{\text{m}}{\text{s}}\times\frac{0.40\text{ m}}{0.4472\text{ m}} = -9.615\frac{\text{m}}{\text{s}} \qquad (4)$$

Note that we have used the fact that $x = r\cos\theta$ and $y = r\sin\theta$ for convenience in Eqs. 3 and 4. Our final results are summarized to three significant digits:

Results:

$$\boxed{r = 0.447\text{ m},\ \ \theta = 0.464\text{ radians},\ \ u_r = 6.71\frac{\text{m}}{\text{s}},\ \ u_\theta = -9.62\frac{\text{m}}{\text{s}}} \qquad (5)$$

We verify our result by calculating the square of the speed in both coordinate systems. In Cartesian coordinates,

$$V^2 = u^2 + v^2 = \left(10.3\frac{\text{m}}{\text{s}}\right)^2 + \left(-5.6\frac{\text{m}}{\text{s}}\right)^2 = 137.5\frac{\text{m}^2}{\text{s}^2} \qquad (6)$$

In cylindrical coordinates,

$$V^2 = u_r^{\,2} + u_\theta^{\,2} = \left(6.708\frac{\text{m}}{\text{s}}\right)^2 + \left(-9.615\frac{\text{m}}{\text{s}}\right)^2 = 137.5\frac{\text{m}^2}{\text{s}^2} \qquad (7)$$

Discussion Such checks of our algebra are always wise.

9-16

Solution We are to transform a given set of Cartesian velocity components into cylindrical velocity components, and identify the flow.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y or r-θ plane.

Analysis We recognize that $r^2 = x^2 + y^2$. We also know that $y = r\sin\theta$ and $x = r\cos\theta$. Using the results of Problem 9-14, the cylindrical velocity components are

u_r component of velocity:

$$\boxed{u_r = u\cos\theta + v\sin\theta = \frac{Cr\sin\theta\cos\theta}{r^2} - \frac{Cr\sin\theta\cos\theta}{r^2} = 0} \qquad (1)$$

u_θ component of velocity::

$$\boxed{u_\theta = -u\sin\theta + v\cos\theta = -\frac{Cr\sin^2\theta}{r^2} - \frac{Cr\cos^2\theta}{r^2} = \frac{-C}{r}} \qquad (2)$$

where we have also used the fact that $\cos^2\theta + \sin^2\theta = 1$. We recognize the velocity components of Eqs. 1 and 2 as those of a **line vortex**.

Discussion The negative sign in Eq. 2 indicates that this vortex is in the *clockwise* direction.

9-17

Solution We are to transform a given set of cylindrical velocity components into Cartesian velocity components.

Analysis We apply the coordinate transformations given in this chapter, along with the results of Problem 9-16,

x component of velocity:
$$u = u_r \cos\theta - u_\theta \sin\theta = \frac{m}{2\pi r}\frac{x}{r} - \frac{\Gamma}{2\pi r}\frac{y}{r} \quad (1)$$

We recognize that $r^2 = x^2 + y^2$. Thus, Eq. 1 becomes

x component of velocity:
$$u = \frac{1}{2\pi\left(x^2+y^2\right)}\left(mx - \Gamma y\right) \quad (2)$$

Similarly,

y component of velocity:
$$\boxed{v = u_r \sin\theta + u_\theta \cos\theta = \frac{m}{2\pi r}\frac{y}{r} + \frac{\Gamma}{2\pi r}\frac{x}{r}} \quad (3)$$

Again recognizing that $r^2 = x^2 + y^2$, Eq. 3 becomes

y component of velocity:
$$\boxed{v = \frac{1}{2\pi\left(x^2+y^2\right)}\left(my + \Gamma x\right)} \quad (4)$$

We verify our result by calculating the square of the speed in both coordinate systems. In Cartesian coordinates,

$$V^2 = u^2 + v^2 = \frac{1}{4\pi^2\left(x^2+y^2\right)^2}\left(m^2x^2 - 2mx\Gamma y + \Gamma^2 y^2\right) + \frac{1}{4\pi^2\left(x^2+y^2\right)^2}\left(m^2y^2 + 2my\Gamma x + \Gamma^2 x^2\right) \quad (5)$$

Two of the terms in Eq. 5 cancel, and we combine the others. After simplification,

Magnitude of velocity squared:
$$\boxed{V^2 = u^2 + v^2 = \frac{1}{4\pi^2\left(x^2+y^2\right)}\left(m^2 + \Gamma^2\right)} \quad (6)$$

We calculate V^2 from the components given in cylindrical coordinates as well,

Magnitude of velocity squared:
$$\boxed{V^2 = u_r^{\,2} + u_\theta^{\,2} = \frac{m^2}{4\pi^2 r^2} + \frac{\Gamma^2}{4\pi^2 r^2} = \frac{m^2+\Gamma^2}{4\pi^2 r^2}} \quad (7)$$

Finally, since $r^2 = x^2 + y^2$, Eqs. 6 and 7 are the same, and **the results are verified**.

Discussion Such checks of our algebra are always wise.

9-18E

Solution We are to transform a given set of Cartesian coordinates and velocity components into cylindrical coordinates and velocity components.

Analysis First we apply the coordinate transformations given in this chapter,

$$x = r\cos\theta = 5.20\ \text{in}\times\cos\left(30.0^\circ\right)\left(\frac{1\ \text{ft}}{12\ \text{in}}\right) = 0.3753\ \text{ft} \qquad (1)$$

and

$$y = r\sin\theta = 5.20\ \text{in}\times\sin\left(30.0^\circ\right)\left(\frac{1\ \text{ft}}{12\ \text{in}}\right) = 0.2167\ \text{ft} \qquad (2)$$

Next we apply the results of a previous problem,

$$u = u_r\cos\theta - u_\theta\sin\theta = (2.06\ \text{ft/s})\times\cos(30.0^\circ) - (4.66\ \text{ft/s})\times\sin(30.0^\circ) = -0.5460\ \text{ft/s} \qquad (3)$$

and

$$v = u_r\sin\theta + u_\theta\cos\theta = (2.06\ \text{ft/s})\times\sin(30.0^\circ) + (4.66\ \text{ft/s})\times\cos(30.0^\circ) = 5.066\ \text{ft/s} \qquad (4)$$

Our final results are summarized to three significant digits:

Results: $x = \mathbf{0.373\ ft}$, $y = \mathbf{0.217\ ft}$, $u = \mathbf{-0.546\ ft/s}$, $v = \mathbf{5.07\ ft/s}$ (5)

We verify our result by calculating the square of the speed in both coordinate systems. In Cartesian coordinates,

$$V^2 = u^2 + v^2 = (-0.5460\ \text{ft/s})^2 + (5.066\ \text{ft/s})^2 = \mathbf{25.96\ ft^2/s^2} \qquad (6)$$

In cylindrical coordinates,

$$V^2 = u_r^2 + u_\theta^2 = (2.06\ \text{ft/s})^2 + (4.66\ \text{ft/s})^2 = \mathbf{25.96\ ft^2/s^2} \qquad (7)$$

Discussion Such checks of our algebra are always wise.

9-19

Solution We are to perform both integrals of the divergence theorem for a given vector and volume, and verify that they are equal.

Analysis We do the volume integral first:

Volume integral:
$$\int_V \vec{\nabla}\cdot\vec{G}dV = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1}\left(\frac{\partial G_x}{\partial x}+\frac{\partial G_y}{\partial y}+\frac{\partial G_z}{\partial z}\right)dzdydx = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1}\left(4z-2y+y\right)dzdydx \qquad (1)$$

The term in parentheses in Eq. 1 reduces to (4z – y), and we integrate this over z first,

$$\int_V \vec{\nabla}\cdot\vec{G}dV = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\left[2z^2-yz\right]_{z=0}^{z=1}dydx = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\left(2-y\right)dydx$$

Then we integrate over y and then over x,

Volume integral:
$$\int_V \vec{\nabla}\cdot\vec{G}dV = \int_{x=0}^{x=1}\left[2y-\frac{y^2}{2}\right]_{y=0}^{y=1}dx = \int_{x=0}^{x=1}\frac{3}{2}dx = \mathbf{\frac{3}{2}} \qquad (2)$$

Next we calculate the surface integral of the divergence theorem. There are six faces of the cube, and unit vector $\vec{n}$ points outward from each face. So, we split the area integral into six parts and sum them. E.g., the right-most face has $\vec{n}$ = (1,0,0), so $\vec{G}\cdot\vec{n}$ = 4xz on this face. The bottom face has $\vec{n}$ = (0,–1,0), so $\vec{G}\cdot\vec{n}$ = y^2 on this face. The surface integral is then

Surface integral:
$$\oint_A \vec{G}\cdot\vec{n}dA = \underbrace{\left[\int_{y=0}^{y=1}\int_{z=0}^{z=1}\left(4xz\right)dzdy\right]_{x=1}}_{\text{Right face}} + \underbrace{\left[\int_{y=0}^{y=1}\int_{z=0}^{z=1}\left(-4xz\right)dzdy\right]_{x=0}}_{\text{Left face}} + \underbrace{\left[\int_{z=0}^{z=1}\int_{x=0}^{x=1}\left(-y^2\right)dxdz\right]_{y=1}}_{\text{Top face}}$$
$$+\underbrace{\left[\int_{z=0}^{z=1}\int_{x=0}^{x=1}\left(y^2\right)dxdz\right]_{y=0}}_{\text{Bottom face}} + \underbrace{\left[\int_{x=0}^{x=1}\int_{y=0}^{y=1}\left(yz\right)dydx\right]_{z=1}}_{\text{Front face}} + \underbrace{\left[\int_{x=0}^{x=1}\int_{y=0}^{y=1}\left(-yz\right)dydx\right]_{z=0}}_{\text{Back face}} \qquad (3)$$

The three integrals on the far right of Eq. 3 are obviously zero. The other three integrals can be obtained carefully,

$$\oint_A \vec{G}\cdot\vec{n}dA = \int_{y=0}^{y=1}\left[2z^2\right]_{z=0}^{z=1}dy + \int_{z=0}^{z=1}\left[-x\right]_{x=0}^{x=1}dz + \int_{x=0}^{x=1}\left[\frac{y^2}{2}\right]_{y=0}^{y=1}dx = \int_{y=0}^{y=1}\left(2\right)dy + \int_{z=0}^{z=1}\left(-1\right)dz + \int_{x=0}^{x=1}\left(\frac{1}{2}\right)dx \qquad (4)$$

The last three integrals of Eq. 4 are trivial. The final result is

Surface integral:
$$\oint_A \vec{G}\cdot\vec{n}dA = 2-1+\frac{1}{2} = \mathbf{\frac{3}{2}} \qquad (5)$$

Since Eq. 2 and Eq. 5 are equal, the divergence theorem works for this case.

Discussion The integration is simple in this example since each face is flat and normal to an axis. In the general case in which the surface is curved, integration is much more difficult, but the divergence theorem always works.

9-20

Solution We are to expand a dot product in Cartesian coordinates and verify it.

Analysis In Cartesian coordinates the del operator is $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ and we let $\vec{G} = G_x\vec{i} + G_y\vec{j} + G_z\vec{k}$. The left hand side of the equation is thus

Left hand side:

$$\begin{aligned}\vec{\nabla}\cdot\left(f\vec{G}\right) &= \frac{\partial\left(fG_x\right)}{\partial x} + \frac{\partial\left(fG_y\right)}{\partial y} + \frac{\partial\left(fG_z\right)}{\partial z} \\ &= G_x\frac{\partial f}{\partial x} + f\frac{\partial G_x}{\partial x} + G_y\frac{\partial f}{\partial y} + f\frac{\partial G_y}{\partial y} + G_z\frac{\partial f}{\partial z} + f\frac{\partial G_z}{\partial z}\end{aligned} \quad (1)$$

The right hand side of the equation is

Right hand side:

$$\begin{aligned}&\vec{G}\cdot\vec{\nabla}f + f\vec{\nabla}\cdot\vec{G} \\ &= \left(G_x\vec{i} + G_y\vec{j} + G_z\vec{k}\right)\cdot\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) + f\left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right) \\ &= G_x\frac{\partial f}{\partial x} + G_y\frac{\partial f}{\partial y} + G_z\frac{\partial f}{\partial z} + f\frac{\partial G_x}{\partial x} + f\frac{\partial G_y}{\partial y} + f\frac{\partial G_z}{\partial z}\end{aligned} \quad (2)$$

Equations 1 and 2 are the same, and **the given equation is verified**.

Discussion The product rule given in this problem was used in this chapter in the derivation of the alternative form of the continuity equation.

Continuity Equation

9-21C

Solution We are to explain why the derivation of the continuity via the divergence theorem is so much less involved than the derivation of the same equation by summation of mass flow rates through each face of an infinitesimal control volume.

Analysis In the derivation using the divergence theorem, we begin with the control volume form of conservation of mass, and simply apply the divergence theorem. The control volume form was already derived in Chap. 5, so **we begin the derivation in this chapter with an established conservation of mass equation**. On the other hand, the alternative derivation is from "scratch" and therefore requires much more algebra.

Discussion The bottom line is that the divergence theorem enables us to quickly convert the control volume form of the conservation law into the differential form.

9-22C

Solution We are to discuss the material derivative of density for the case of compressible and incompressible flow.

Analysis If the flow field is compressible, we expect that as a fluid particle (a *material element*) moves around in the flow, its density changes. Thus the *material derivative of density* (the rate of change of density following a fluid particle) is **non-zero for compressible flow**. However, if the flow field is incompressible, the density remains constant. As a fluid particle moves around in the flow, the material derivative of density must be **zero for incompressible flow** (no change in density following the fluid particle).

Discussion The material derivative of any property is the rate of change of that property following a fluid particle.

9-23

Solution We are to repeat Example 9-1, but without using continuity.

Assumptions **1** Density varies with time, but not space; in other words, the density is uniform throughout the cylinder at any given time, but changes with time. **2** No mass escapes from the cylinder during the compression.

Analysis The mass inside the cylinder is constant, but the volume decreases linearly as the piston moves up. At $t = 0$ when $L = L_{\text{Bottom}}$ the initial volume of the cylinder is $V(0) = L_{\text{Bottom}}A$, where A is the cross-sectional area of the cylinder. At $t = 0$ the density is $\rho = \rho(0) = m/V(0)$, and thus

Mass in the cylinder:

$$m = \rho(0)V(0) = \rho(0)L_{\text{Bottom}}A \qquad (1)$$

Mass m (Eq. 1) is a constant since no mass escapes during the compression. At some later time t, $L = L_{\text{Bottom}} - V_{\text{P}}t$ and the volume is thus

Cylinder volume at time t:

$$V = (L_{\text{Bottom}} - V_{\text{P}}t)A \qquad (2)$$

The density at time t is

Density at time t:

$$\rho = \frac{m}{V} = \frac{\rho(0)L_{\text{Bottom}}A}{(L_{\text{Bottom}} - V_{\text{P}}t)A} \qquad (3)$$

where we have plugged in Eq. 1 for m and Eq. 2 for V. Equation 3 reduces to

$$\rho = \rho(0)\frac{L_{\text{Bottom}}}{L_{\text{Bottom}} - V_{\text{P}}t} \qquad (4)$$

or, using the nondimensional variables of Example 9-1,

Nondimensional result:

$$\boxed{\frac{\rho}{\rho(0)} = \frac{1}{1 - \dfrac{V_{\text{P}}t}{L_{\text{Bottom}}}} \quad \text{or} \quad \rho^* = \frac{1}{1 - t^*}} \qquad (5)$$

which is identical to Eq. 5 of Example 9-1.

Discussion We see by this exercise that the continuity equation is indeed an equation of conservation of mass.

9-24

Solution We are to expand the continuity equation in Cartesian coordinates.

Analysis We expand the second term by taking the dot product of the del operator $\vec{\nabla} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$ with $\rho\vec{V} = (\rho u)\vec{i} + (\rho v)\vec{j} + (\rho w)\vec{k}$, giving

Compressible continuity equation in Cartesian coordinates:

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0} \qquad (1)$$

We can further expand Eq. 1 by using the product rule on the spatial derivatives, resulting in 7 terms,

Further expansion:

$$\boxed{\frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial\rho}{\partial y} + \rho\frac{\partial w}{\partial z} + w\frac{\partial\rho}{\partial z} = 0} \qquad (2)$$

Discussion We can do a similar thing in cylindrical coordinates, but the algebra is somewhat more complicated.

9-25

Solution We are to write the given equation as a word equation and discuss it.

Analysis Here is a word equation: "**The time rate of change of volume of a fluid particle per unit volume is equal to the divergence of the velocity field**." As a fluid particle moves around in a compressible flow, it can distort, rotate, and get larger or smaller. Thus the volume of the fluid element can change with time; this is represented by the left hand side of the equation. The right hand side is identically zero for an incompressible flow, but it is *not* zero for a compressible flow. Thus we can think of the volumetric strain rate as a measure of compressibility of a fluid flow.

Discussion Volumetric strain rate is a kinematic property as discussed in Chap. 4. Nevertheless, it is shown here to be related to the continuity equation (conservation of mass).

9-26

Solution We are to verify that a given flow field satisfies the continuity equation, and we are to discuss conservation of mass at the origin.

Analysis The 2-D cylindrical velocity components (u_r,u_θ) for this flow field are

Cylindrical velocity components:

$$u_r = \frac{m}{2\pi r} \qquad u_\theta = \frac{\Gamma}{2\pi r} \qquad (1)$$

where m and Γ are constants We plug Eq. 1 into the incompressible continuity equation in cylindrical coordinates,

Incompressible continuity:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta} + \frac{\partial(u_z)}{\partial z} = 0 \quad \text{or} \quad \underbrace{\cancel{\frac{1}{r}\frac{\partial\left(\frac{m}{2\pi}\right)}{\partial r}}}_{0} + \underbrace{\cancel{\frac{1}{r}\frac{\partial\left(\frac{\Gamma}{2\pi r}\right)}{\partial \theta}}}_{0} + \underbrace{\cancel{\frac{\partial(u_z)}{\partial z}}}_{0} = 0 \qquad (2)$$

The first term is zero because it is the derivative of a constant. The second term is zero because r is not a function of θ. The third term is zero since this is a 2-D flow with $u_z = 0$. Thus, we verify that *the incompressible continuity equation is satisfied for the given velocity field*.

At the origin, both u_r and u_θ go to infinity. Conservation of mass is not affected by u_θ, but the fact that u_r is non-zero at the origin violates conservation of mass. We think of the flow along the z axis as a *line sink* toward which mass approaches from all directions in the plane and then disappears (like a black hole in two dimensions). *Mass is not conserved at the origin*.

Discussion Singularities such as this are unphysical of course, but are nevertheless useful as approximations of real flows, as long as we stay away from the singularity itself.

9-27

Solution We are to verify that a given velocity field satisfies continuity.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis The velocity field of Problem 9-16 is

Cartesian velocity components:
$$u = \frac{Cy}{x^2 + y^2} \qquad v = \frac{-Cx}{x^2 + y^2} \qquad (1)$$

We check continuity, staying in Cartesian coordinates,

$$\underbrace{\frac{\partial u}{\partial x}}_{-2xCy\left(x^2+y^2\right)^{-3}} + \underbrace{\frac{\partial v}{\partial y}}_{2yCx\left(x^2+y^2\right)^{-3}} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0 \text{ since 2-D}} = 0$$

So we see that **the incompressible continuity equation is indeed satisfied**.

Discussion The fact that the flow field satisfies continuity does not guarantee that a corresponding pressure field exists that can satisfy the steady conservation of momentum equation. In this case, however, it does.

9-28

Solution We are to verify that a given velocity field is incompressible.

Assumptions **1** The flow is two-dimensional, implying no *z component* of velocity and no variation of *u* or *v* with *z*.

Analysis The components of velocity in the *x* and *y* directions respectively are

$$u = 1.6 + 1.8x \qquad v = 1.5 - 1.8y$$

To check if the flow is incompressible, we see if the incompressible continuity equation is satisfied:

$$\underbrace{\frac{\partial u}{\partial x}}_{2.8} + \underbrace{\frac{\partial v}{\partial y}}_{-2.8} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0 \text{ since 2-D}} = 0 \qquad \text{or} \qquad 1.8 - 1.8 = 0$$

So we see that **the incompressible continuity equation is indeed satisfied**. Hence the flow field is incompressible.

Discussion The fact that the flow field satisfies continuity does not guarantee that a corresponding pressure field exists that can satisfy the steady conservation of momentum equation.

9-29

Solution For a given axial velocity component in an axisymmetric flow field, we are to generate the radial velocity component.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is axisymmetric implying that $u_\theta = 0$ and there is no variation in the θ direction.

Analysis We use the incompressible continuity equation in cylindrical coordinates, simplified as follows for axisymmetric flow,

Incompressible axisymmetric continuity equation:
$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\partial(u_z)}{\partial z}=0 \qquad (1)$$

We rearrange Eq. 1,

$$\frac{\partial(ru_r)}{\partial r}=-r\frac{\partial(u_z)}{\partial z}=-r\frac{u_{z,\text{exit}}-u_{z,\text{entrance}}}{L} \qquad (2)$$

We integrate Eq. 2 with respect to r,

$$ru_r=-\frac{r^2}{2}\frac{u_{z,\text{exit}}-u_{z,\text{entrance}}}{L}+f(z) \qquad (3)$$

Notice that since we performed a *partial* integration with respect to r, we add a function of the other variable z rather than simply a constant of integration. We divide all terms in Eq. 3 by r and recognize that the term with $f(z)$ will go to infinity at the centerline of the nozzle ($r = 0$) unless $f(z) = 0$. We write our final expression for u_r,

Radial velocity component:
$$\boxed{u_r=-\frac{r}{2}\frac{u_{z,\text{exit}}-u_{z,\text{entrance}}}{L}} \qquad (4)$$

Discussion You should plug the given equation and Eq. 4 into Eq. 1 to verify that the result is correct. (It is.)

9-30

Solution We are to determine a relationship between constants a, b, c, and d that ensures incompressibility.

Assumptions **1** The flow is steady. **2** The flow is incompressible (under certain restraints to be determined).

Analysis We plug the given velocity components into the incompressible continuity equation,

Condition for incompressibility:
$$\underbrace{\frac{\partial u}{\partial x}}_{ay^2}+\underbrace{\frac{\partial v}{\partial y}}_{-6cy^2}+\underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0}=0 \qquad ay^2-6cy^2=0$$

Thus to guarantee incompressibility, constants a and c must satisfy the following relationship:

Condition for incompressibility:
$$\boxed{a=6c} \qquad (1)$$

Discussion If Eq. 1 were not satisfied, the given velocity field might still represent a valid flow field, but density would have to vary with location in the flow field – in other words the flow would be *compressible*.

9-31

Solution We are to determine a relationship between constants a, b, c, and d that ensures incompressibility.

Assumptions **1** The flow is steady. **2** The flow is incompressible (under certain restraints to be determined).

Analysis We plug the given velocity components into the incompressible continuity equation,

Condition for incompressibility:
$$\underbrace{\frac{\partial u}{\partial x}}_{2axy}+\underbrace{\frac{\partial v}{\partial y}}_{2cxy}+\underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0}=0 \qquad 2axy+2cxy=0$$

Thus to guarantee incompressibility, constants a and c must satisfy the following relationship:

Condition for incompressibility:
$$\boxed{a=-c} \qquad (1)$$

Discussion If Eq. 1 were not satisfied, the given velocity field might still represent a valid flow field, but density would have to vary with location in the flow field – in other words the flow would be *compressible*.

9-32

Solution We are to find the *y component* of velocity, v, using a given expression for u.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y plane, implying that $w = 0$ and neither u nor v depend on z.

Analysis Since the flow is steady and incompressible, we apply the incompressible continuity in Cartesian coordinates to the flow field, giving

Condition for incompressibility:
$$\frac{\partial v}{\partial y}=-\underbrace{\frac{\partial u}{\partial x}}_{a}-\underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0} \qquad \frac{\partial v}{\partial y}=-a$$

Next we integrate with respect to y. Note that since the integration is a *partial* integration, we must add some arbitrary function of x instead of simply a constant of integration.

Solution:
$$\boxed{v=-ay+f\left(x\right)}$$

If the flow were three-dimensional, we would add a function of x and z instead.

Discussion To satisfy the incompressible continuity equation, any function of x will work since there are no derivatives of v with respect to x in the continuity equation. Not all functions of x are necessarily physically possible, however, since the flow must also satisfy the steady conservation of momentum equation.

9-33

Solution We are to find the most general form of the tangential velocity component of a purely circular flow that does not violate conservation of mass.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* or *r-θ* plane.

Analysis We use cylindrical coordinates for convenience. We solve for u_θ using the incompressible continuity equation,

$$\underbrace{\frac{1}{\cancel{r}}\frac{\partial\left(\cancel{ru_r}\right)}{\partial r}}_{\text{0 for circular flow}}+\frac{1}{r}\frac{\partial\left(u_\theta\right)}{\partial\theta}+\underbrace{\frac{\partial\left(\cancel{u_z}\right)}{\cancel{\partial z}}}_{\text{0 for 2-D flow}}=0 \quad \text{or} \quad \frac{\partial\left(u_\theta\right)}{\partial\theta}=0 \qquad (1)$$

We integrate Eq. 1 with respect to θ, adding a function of the other variable r rather than simply a constant of integration since this is a partial integration,

Result:

$$\boxed{u_\theta=f\left(r\right)} \qquad (2)$$

Discussion Any function of r in Eq. 2 will satisfy the continuity equation.

9-34

Solution We are to find the *y component* of velocity, v, using a given expression for u.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* plane, implying that $w = 0$ and neither u nor v depend on z.

Analysis We plug the velocity components into the steady incompressible continuity equation,

Condition for incompressibility:

$$\frac{\partial v}{\partial y}=-\underbrace{\frac{\partial u}{\partial x}}_{a}-\underbrace{\frac{\partial \cancel{w}}{\cancel{\partial z}}}_{0} \qquad \frac{\partial v}{\partial y}=-a$$

Next we integrate with respect to y. Note that since the integration is a *partial* integration, we must add some arbitrary function of x instead of simply a constant of integration.

Solution:

$$\boxed{v=-ay+f\left(x\right)}$$

If the flow were three-dimensional, we would add a function of x and z instead.

Discussion To satisfy the incompressible continuity equation, any function of x will work since there are no derivatives of v with respect to x in the continuity equation. Not all functions of x are necessarily physically possible, however, since the flow may not be able to satisfy the steady conservation of momentum equation.

9-35

Solution We are to find the *y component* of velocity, v, using a given expression for u.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y plane, implying that $w = 0$ and neither u nor v depend on z.

Analysis We plug the velocity components into the steady incompressible continuity equation,

Condition for incompressibility: $$\frac{\partial v}{\partial y} = -\underbrace{\frac{\partial u}{\partial x}}_{6ax-2by} - \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0} \qquad \frac{\partial v}{\partial y} = -6ax + 2by$$

Next we integrate with respect to y. Note that since the integration is a *partial* integration, we must add some arbitrary function of x instead of simply a constant of integration.

Solution: $$\boxed{v = -6axy + by^2 + f(x)}$$

If the flow were three-dimensional, we would add a function of x and z instead.

Discussion To satisfy the incompressible continuity equation, any function of x will work since there are no derivatives of v with respect to x in the continuity equation.

9-36

Solution We are to find the most general form of the radial velocity component of a purely radial flow that does not violate conservation of mass.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y or r-θ plane.

Analysis We use cylindrical coordinates for convenience. We solve for u_r using the incompressible continuity equation,

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \underbrace{\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}}_{\text{0 for radial flow}} + \underbrace{\cancel{\frac{\partial(u_z)}{\partial z}}}_{\text{0 for 2-D flow}} = 0 \quad \text{or} \quad \frac{\partial(ru_r)}{\partial r} = 0 \qquad (1)$$

We integrate Eq. 1 with respect to r, adding a function of the other variable θ rather than simply a constant of integration since this is a partial integration,

Result: $$\boxed{ru_r = f(\theta) \quad \text{or} \quad u_r = \frac{f(\theta)}{r}} \qquad (2)$$

Discussion Any function of θ in Eq. 2 will satisfy the continuity equation.

9-37

Solution We are to find the *z component* of velocity using given expressions for *u* and *v*.

Assumptions **1** The flow is steady. **2** The flow is incompressible.

Analysis We apply the steady incompressible continuity equation to the given flow field,

Condition for incompressibility: $$\frac{\partial w}{\partial z} = -\underbrace{\frac{\partial u}{\partial x}}_{2a+by} - \underbrace{\frac{\partial v}{\partial y}}_{-bz^2} \qquad \frac{\partial w}{\partial z} = -2a - by + bz^2$$

Next we integrate with respect to z. Note that since the integration is a *partial* integration, we must add some arbitrary function of x and y instead of simply a constant of integration.

Solution: $$\boxed{w = -2az - byz + \frac{bz^3}{3} + f\left(x, y\right)}$$

Discussion To satisfy the incompressible continuity equation, any function of x and y will work since there are no derivatives of w with respect to x or y in the continuity equation.

9-38

Solution For given velocity component u and density ρ, we are to predict velocity component v, plot an approximate shape of the duct, and predict its height at section (2).

Assumptions **1** The flow is steady and two-dimensional in the x-y plane, but *compressible*. **2** Friction on the walls is ignored. **3** Axial velocity u and density ρ vary linearly with x. **4** The x axis is a line of top-bottom symmetry.

Properties The fluid is standard air. The speed of sound is about 340 m/s, so the flow is subsonic, but compressible.

Analysis (a) We write expressions for u and ρ, forcing them to be linear in x,

$$u = u_1 + C_u x \qquad C_u = \frac{u_2 - u_1}{\Delta x} = \frac{(100-300)\dfrac{\text{m}}{\text{s}}}{2.0\text{ m}} = -100\frac{1}{\text{s}} \tag{1}$$

$$\rho = \rho_1 + C_\rho x \qquad C_\rho = \frac{\rho_2 - \rho_1}{\Delta x} = \frac{(1.2-0.85)\dfrac{\text{kg}}{\text{m}^3}}{2.0\text{ m}} = 0.175\frac{\text{kg}}{\text{m}^4} \tag{2}$$

where C_u and C_ρ are constants. We use the compressible form of the steady continuity equation, placing the unknown term v on the left hand side, and plugging in Eqs. 1 and 2,

$$\frac{\partial(\rho v)}{\partial y} = -\frac{\partial(\rho u)}{\partial x} = -\frac{\partial\left((\rho_1 + C_\rho x)(u_1 + C_u x)\right)}{\partial x}$$

After some algebra,

$$\frac{\partial(\rho v)}{\partial y} = -\left(\rho_1 C_u + u_1 C_\rho\right) - 2C_u C_\rho x \tag{3}$$

We integrate Eq. 3 with respect to y,

$$\rho v = -\left(\rho_1 C_u + u_1 C_\rho\right)y - 2C_u C_\rho xy + f(x) \tag{4}$$

Since this is a *partial* integration, we add an arbitrary function of x instead of simply a constant of integration. We now apply boundary conditions. Since the flow is symmetric about the x axis (y = 0), v must equal zero at y = 0 for any x. This is possible only if $f(x)$ is identically zero. Applying $f(x)$ = 0, dividing by ρ to solve for v, and plugging in Eq. 2, Eq. 4 becomes

$$\boxed{v = \frac{-\left(\rho_1 C_u + u_1 C_\rho\right)y - 2C_u C_\rho xy}{\rho} = \frac{-\left(\rho_1 C_u + u_1 C_\rho\right)y - 2C_u C_\rho xy}{\rho_1 + C_\rho x}} \tag{5}$$

(b) For known values of u and v, we can plot streamlines between x = 0 and x = 2.0 m using the technique described in Chap. 4. Several streamlines are shown in Fig. 1. The streamline starting at x = 0, y = 0.8 m is the top wall of the duct.

(c) At section (2), the top streamline crosses y = 1.70 m at x = 2.0 m. Thus, *the predicted height of the duct at section* (2) *is* 1.70 m.

Discussion You can verify that the combination of Eqs. 1, 2, and 5 satisfies the steady compressible continuity equation. However, this alone does not guarantee that the density and velocity components will actually *follow* these equations if this diverging duct were to be built. The actual flow depends on the *pressure rise* between sections (1) and (2) – only one unique pressure rise can yield the desired flow deceleration. Temperature may also change considerably in this kind of compressible flow field.

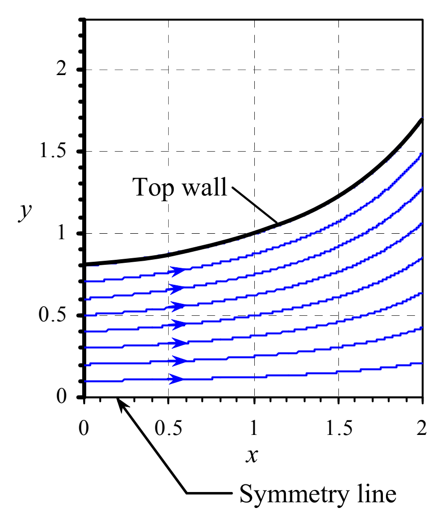


FIGURE 1
Streamlines for a diverging duct.

Stream Function

9-39C

Solution We are to discuss the significance of the difference in value of stream function from one streamline to another.

Analysis **The difference in the value of ψ from one streamline to another is equal to the volume flow rate per unit width between the two streamlines**.

Discussion This fact about the stream function can be used to calculate the volume flow rate in certain applications.

9-40C

Solution We are to discuss why the stream function is called a non-primitive variable in CFD lingo.

Analysis The natural physical variables in a fluid flow problem are the velocity components and the pressure. [If the flow is compressible, density and temperature are also natural physical variables.] These variables can be considered "primitive" because we do not change them in any way – we simply solve for them directly. Stream function, on the other hand, is a contrived or derived variable. **The stream function is not primitive in the sense that it is not one of the original physical variables in the problem**.

Discussion Vorticity is another example of a non-primitive variable. In fact, some 2-D CFD codes use stream function and vorticity as the variables – non-primitive variables.

9-41C

Solution We are to discuss the restrictions on the stream function that cause it to exactly satisfy 2-D incompressible continuity, and why they are necessary.

Analysis **Stream function ψ must be a *smooth function* of x and y (or r and θ)**. These restrictions are necessary so that the second derivatives of ψ with respect to both variables are equal regardless of the order of differentiation. In other words, if $\frac{\partial^2\psi}{\partial x\partial y}=\frac{\partial^2\psi}{\partial y\partial x}$, then the 2-D incompressible continuity equation is satisfied exactly by the definition of ψ.

Discussion If the stream function were not smooth, there would be sudden discontinuities in the velocity field as well – a physical impossibility that would violate conservation of mass.

9-42C

Solution We are to discuss the significance of curves of constant stream function, and why the stream function is useful.

Analysis Curves of constant stream function represent streamlines of a flow. A stream function is useful because by drawing curves of constant ψ, we can visualize the instantaneous velocity field. In addition, the change in the value of ψ from one streamline to another is equal to the volume flow rate per unit width between the two streamlines.

Discussion Streamlines are an instantaneous flow description, as discussed in Chap. 4.

9-43

Solution For a given velocity field we are to generate an expression for ψ, and we are to calculate the volume flow rate per unit width between two streamlines.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis We start by picking one of the two definitions of the stream function (it doesn't matter which part we choose – the solution will be identical).

$$\frac{\partial \psi}{\partial y} = u = V \qquad (1)$$

Next we integrate Eq. 1 with respect to y, noting that this is a *partial* integration and we must add an arbitrary function of the other variable, x, rather than a simple constant of integration.

$$\psi = Vy + g(x) \qquad (2)$$

Now we choose the other part of the definition of ψ, differentiate Eq. 2, and rearrange as follows:

$$v = -\frac{\partial \psi}{\partial x} = -g'(x) \qquad (3)$$

where $g'(x)$ denotes dg/dx since g is a function of only one variable, x. We now have two expressions for velocity component v, the given equation and Eq. 3. We equate these and integrate with respect to x to find $g(x)$,

$$v = 0 = -g'(x) \qquad g'(x) = 0 \qquad g(x) = C \qquad (4)$$

Note that here we have added an arbitrary constant of integration C since g is a function of x only. Finally, plugging Eq. 4 into Eq. 2 yields the final expression for ψ,

Stream function: $$\boxed{\psi = Vy + C} \qquad (5)$$

Constant C is arbitrary; it is common to set it to zero, although it can be set to any desired value. Here, $\psi = 0$ along the streamline at $y = 0$, forcing C to equal zero by Eq. 5. For the streamline at $y = 0.5$ m,

Value of ψ_2: $$\psi_2 = \left(6.94\ \frac{\text{m}}{\text{s}}\right) \times (0.5\ \text{m}) = 3.47\frac{\text{m}^2}{\text{s}} \qquad (6)$$

The volume flow rate per unit width between streamlines ψ_2 and ψ_0 is equal to $\psi_2 - \psi_0$,

Volume flow rate per unit width: $$\frac{\dot{V}}{W} = \psi_2 - \psi_0 = (3.47 - 0)\frac{\text{m}^2}{\text{s}} = \mathbf{3.47}\frac{\mathbf{m^2}}{\mathbf{s}} \qquad (7)$$

We verify our result by calculating the volume flow rate per unit width from first principles. Namely, volume flow rate is equal to speed times cross-sectional area,

Volume flow rate per unit width:

$$\frac{\dot{V}}{W} = V(y_2 - y_0) = 6.94\frac{\text{m}}{\text{s}} \times (0.5 - 0)\ \text{m} = \mathbf{3.47}\frac{\mathbf{m^2}}{\mathbf{s}} \qquad (8)$$

Discussion If constant C were some value besides zero, we would still get the same result for the volume flow rate since C would cancel out in the subtraction.

9-44

Solution For a given velocity field we are to show that the velocity field satisfies the continuity equation, and we are to determine the stream function corresponding to this velocity field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional.

Analysis For a two-dimensional flow, the continuity equation in cylindrical coordinates is, from Eq. 9–18,

$$\frac{\partial(ru_r)}{\partial r}+\frac{\partial(u_\theta)}{\partial\theta}=0$$

or

$$\frac{\partial}{\partial r}\left(rU_\infty\left(1-\frac{R^2}{r^2}\right)\cos\theta\right)+\frac{\partial}{\partial\theta}\left(-rU_\infty\left(1+\frac{R^2}{r^2}\right)\sin\theta\right)$$
$$=\underbrace{\left[U_\infty\left(1-\frac{R^2}{r^2}\right)\cos\theta+\frac{2U_\infty R^2}{r^2}\cos\theta\right]}_{U_\infty\left(1+\frac{R^2}{r^2}\right)\cos\theta}-U_\infty\left(1+\frac{R^2}{r^2}\right)\cos\theta=0$$

Therefore the velocity field satisfies the continuity equation. The stream function can be determined from Eq. 9–27 as follows:

$$u_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta}\rightarrow\psi(r,\theta)=\int rU_\infty\left(1-\frac{R^2}{r^2}\right)\cos\theta\,d\theta=rU_\infty\left(1-\frac{R^2}{r^2}\right)\sin\theta+f(r)$$

$$u_\theta=-\frac{\partial\psi}{\partial r}\rightarrow\psi(r,\theta)=-\int\left[-U_\infty\left(1+\frac{R^2}{r^2}\right)\sin\theta\right]dr=rU_\infty\left(1-\frac{R^2}{r^2}\right)\sin\theta+g(\theta)$$

Therefore we see that the stream function is

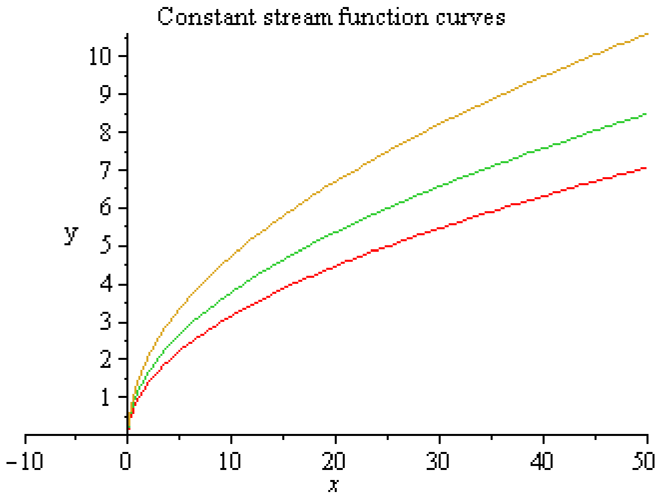
$$\psi(r,\theta)=rU_\infty\left(1-\frac{R^2}{r^2}\right)\sin\theta$$

9-45

Solution For a given stream function we are to sketch stremalines, derive expressions for the velocity components, and determine the pathlines at $t = 0$.

Assumptions **1** The flow is unsteady. **2** The flow is incompressible. **3** The flow is two-dimensional.

Analysis The streamlines are shown below for different values of stream function.



The velocity component can be found from Eq. 9–20 as follows:

$$u = \frac{\partial \psi}{\partial y} = -\frac{8x}{y^3}t$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{4}{y^2}t$$

The pathlines are determined from the relations

$$u = \frac{dx}{dt} \text{ and } \quad v = \frac{dy}{dt}$$

from which we obtain

$$-\frac{8x}{y^3}t = \frac{dx}{dt} \to \int_{x_0}^{x} \frac{dx}{x} = -\frac{8}{y^3}\int_0^t t\,dt \to \ln\frac{x}{x_0} = -\frac{4}{y^3}t^2 \to x = x_0 \exp\left(-\frac{4}{y^3}t^2\right)$$

$$-\frac{4}{y^2}t = \frac{dy}{dt} \to \int_{y_0}^{y} y^2 dy = -4\int_0^t t\,dt \to \frac{1}{3}(y^3 - y_0^3) = -2t^2 \to y = \sqrt[3]{y_0^3 - 6t^2}$$

9-46

Solution We are to generate an expression for the stream function along a vertical line in a given flow field, and we are to determine ψ at the top wall.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** The flow is fully developed.

Analysis We start by picking one of the two definitions of the stream function (it doesn't matter which part we choose – the solution will be identical).

$$\frac{\partial \psi}{\partial y} = u = \frac{V}{h} y \qquad (1)$$

Next we integrate Eq. 1 with respect to *y*, noting that this is a *partial* integration and we must add an arbitrary function of the other variable, *x*, rather than a simple constant of integration.

$$\psi = \frac{V}{2h} y^2 + g(x) \qquad (2)$$

Now we choose the other part of the definition of ψ, differentiate Eq. 2, and rearrange as follows:

$$v = -\frac{\partial \psi}{\partial x} = -g'(x) \qquad (3)$$

where $g'(x)$ denotes dg/dx since *g* is a function of only one variable, *x*. We now have two expressions for velocity component *v*, the given equation and Eq. 3. We equate these and integrate with respect to *x*, we find $g(x)$,

$$v = 0 = -g'(x) \qquad g'(x) = 0 \qquad g(x) = C \qquad (4)$$

Note that here we have added an arbitrary constant of integration *C* since *g* is a function of *x* only. Finally, plugging Eq. 4 into Eq. 2 yields the final expression for ψ,

Stream function:

$$\psi = \frac{V}{2h} y^2 + C \qquad (5)$$

We find constant *C* by employing the boundary condition on ψ. Here, $\psi = 0$ along $y = 0$ (the bottom wall). Thus *C* is equal to zero by Eq. 5, and

Stream function:

$$\boxed{\psi = \frac{V}{2h} y^2} \qquad (6)$$

Along the top wall, $y = h$, and thus

Stream function along top wall:

$$\boxed{\psi_{\text{top}} = \frac{V}{2h} h^2 = \frac{Vh}{2}} \qquad (7)$$

Discussion The stream function of Eq. 6 is valid not only along the vertical dashed line of the figure provided in the problem statement, but *everywhere* in the flow since the flow is fully developed and there is nothing special about any particular *x* location.

9-47

Solution We are to generate an expression for the volume flow rate per unit width for Couette flow. We are to compare results from two methods of calculation.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y plane. **4** The flow is fully developed.

Analysis We integrate the x component of velocity times cross-sectional area to obtain volume flow rate,

$$\dot{V} = \int_A u dA = \int_{y=0}^{y=h} \frac{V}{h} yW dy = \left[\frac{Vy^2}{2h} W \right]_{y=0}^{y=h} = \frac{Vh}{2} W \qquad (1)$$

where W is the width of the channel into the page. On a per unit width basis, we divide Eq. 1 by W to get

Volume flow rate per unit width: $$\boxed{\frac{\dot{V}}{W} = \frac{Vh}{2}} \qquad (2)$$

The volume flow rate per unit width between any two streamlines ψ_2 and ψ_1 is equal to $\psi_2 - \psi_1$. We take the streamlines representing the top wall and the bottom wall of the channel. Using the result from the previous problem,

Volume flow rate per unit width: $$\boxed{\frac{\dot{V}}{W} = \psi_{\text{top}} - \psi_{\text{bottom}} = \frac{Vh}{2} - 0 = \frac{Vh}{2}} \qquad (3)$$

Equations 2 and 3 agree, as they must.

Discussion The integration of Eq. 1 can be performed at any x location in the channel since the flow is fully developed.

9-48E

Solution We are to plot several streamlines using evenly spaced values of ψ and discuss the spacing between the streamlines.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** The flow is fully developed.

Analysis The stream function is obtained from the result of Problem 9-40,

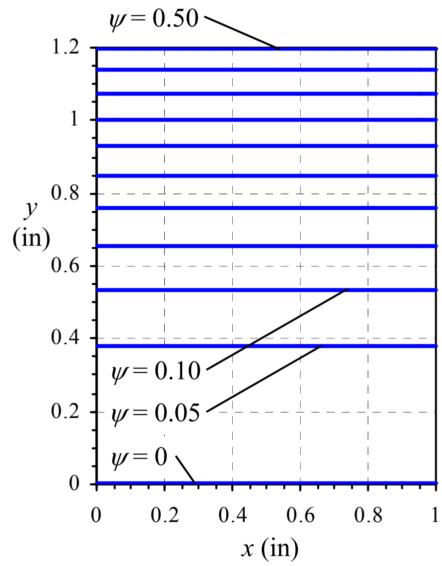
Stream function: $$\psi = \frac{V}{2h}y^2 \qquad (1)$$

We solve Eq. 1 for y as a function of ψ so that we can plot streamlines,

Equation for streamlines: $$y = \sqrt{\frac{2h\psi}{V}} \qquad (2)$$

We have taken only the positive root in Eq. 2 for obvious reasons. Along the top wall, $y = h$, and thus

$$\psi_{\text{top}} = \frac{Vh}{2} = \frac{10.0\,\frac{\text{ft}}{\text{s}} \times 0.100\text{ ft}}{2} = 0.500\,\frac{\text{ft}^2}{\text{s}} \qquad (3)$$



The streamlines themselves are straight, flat horizontal lines as seen by Eq. 1. We divide ψ_{top} by 10 to generate evenly spaced stream functions. We plot 11 streamlines in the figure (counting the streamlines on both walls) by plugging these values of ψ into Eq. 2. **The streamlines are not evenly spaced**. This is because the volume flow rate per unit width between two streamlines ψ_2 and ψ_1 is equal to $\psi_2 - \psi_1$. The flow speeds near the top of the channel are higher than those near the bottom of the channel, so we expect the streamlines to be closer near the top.

Discussion The extent of the x axis in the figure is arbitrary since the flow is fully developed. You can immediately see from a streamline plot like Fig. 1 where flow speeds are high and low (relatively speaking).

9-49

Solution We are to generate an expression for the stream function along a vertical line in a given flow field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** The flow is fully developed.

Analysis We start by picking one of the two definitions of the stream function (it doesn't matter which part we choose – the solution will be identical).

$$\frac{\partial \psi}{\partial y} = u = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right) \quad (1)$$

Next we integrate Eq. 1 with respect to y, noting that this is a *partial* integration and we must add an arbitrary function of the other variable, x, rather than a simple constant of integration.

$$\psi = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{y^3}{3} - h\frac{y^2}{2}\right) + g(x) \quad (2)$$

Now we choose the other part of the definition of ψ, differentiate Eq. 2, and rearrange as follows:

$$v = -\frac{\partial \psi}{\partial x} = -g'(x) \quad (3)$$

where $g'(x)$ denotes dg/dx since g is a function of only one variable, x. We now have two expressions for velocity component v, the given equation and Eq. 3. We equate these and integrate with respect to x to find $g(x)$,

$$v = 0 = -g'(x) \qquad g'(x) = 0 \qquad g(x) = C \quad (4)$$

Note that here we have added an arbitrary constant of integration C since g is a function of x only. Finally, plugging Eq. 4 into Eq. 2 yields the final expression for ψ,

Stream function:
$$\psi = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{y^3}{3} - h\frac{y^2}{2}\right) + C \quad (5)$$

We find constant C by employing the boundary condition on ψ. Here, $\psi = 0$ along $y = 0$ (the bottom wall). Thus C is equal to zero by Eq. 5, and

Stream function:
$$\boxed{\psi = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{y^3}{3} - h\frac{y^2}{2}\right)} \quad (6)$$

Along the top wall, $y = h$, and thus

Stream function along top wall:
$$\boxed{\psi_{\text{top}} = -\frac{1}{12\mu}\frac{dP}{dx}h^3} \quad (7)$$

Discussion The stream function of Eq. 6 is valid not only along the vertical dashed line of the figure provided in the problem statement, but *everywhere* in the flow since the flow is fully developed and there is nothing special about any particular x location.

9-50

Solution We are to generate an expression for the volume flow rate per unit width for fully developed channel flow. We are to compare results from two methods of calculation.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** The flow is fully developed.

Analysis We integrate the *x* component of velocity times cross-sectional area to obtain volume flow rate,

$$\dot{V} = \int_A u dA = \int_{y=0}^{y=h} \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)W dy = \left[\frac{1}{2\mu}\frac{dP}{dx}\left(\frac{y^3}{3} - h\frac{y^2}{2}\right)W\right]_{y=0}^{y=h}$$
$$= \frac{1}{2\mu}\frac{dP}{dx}\left(-\frac{h^3}{6}\right)W = -\frac{1}{12\mu}\frac{dP}{dx}h^3 W \qquad (1)$$

where W is the width of the channel into the page. On a per unit width basis, we divide Eq. 1 by W to get

Volume flow rate per unit width:
$$\boxed{\frac{\dot{V}}{W} = -\frac{1}{12\mu}\frac{dP}{dx}h^3} \qquad (2)$$

The volume flow rate per unit width between any two streamlines ψ_2 and ψ_1 is equal to $\psi_2 - \psi_1$. We take the streamlines representing the top wall and the bottom wall of the channel. Using the result from the previous problem,

Volume flow rate per unit width:
$$\boxed{\frac{\dot{V}}{W} = \psi_{\text{top}} - \psi_{\text{bottom}} = -\frac{1}{12\mu}\frac{dP}{dx}h^3 - 0 = -\frac{1}{12\mu}\frac{dP}{dx}h^3} \qquad (3)$$

Equations 2 and 3 agree, as they must.

Discussion The integration of Eq. 1 can be performed at any *x* location in the channel since the flow is fully developed.

9-51

Solution We are to plot several streamlines using evenly spaced values of ψ and discuss the spacing between the streamlines.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y plane. **4** The flow is fully developed.

Properties The viscosity of water at T = 20°C is 1.002 × 10^{-3} kg/(m·s).

Analysis The stream function is obtained from the result of Problem 9-43,

Stream function: $$\psi = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{y^3}{3} - h\frac{y^2}{2}\right) \qquad (1)$$

We need to solve Eq. 1 (a cubic equation) for y as a function of ψ so that we can plot streamlines. First we re-write Eq. 1 in standard cubic form,

Standard cubic form: $$y^3 - \frac{3h}{2}y^2 - \frac{6\mu\psi}{dP/dx} = 0 \qquad (2)$$

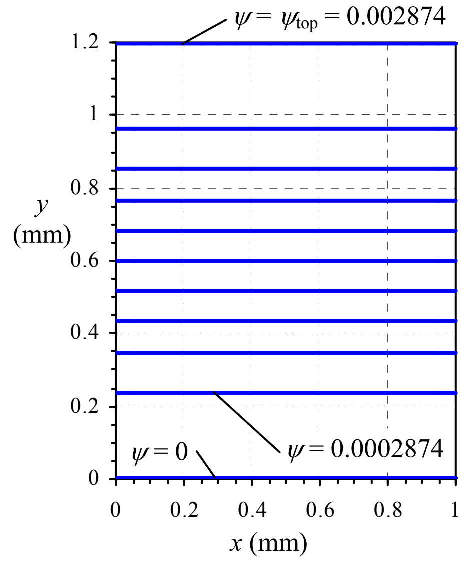


FIGURE 1
Streamlines for 2-D channel flow with evenly spaced values of stream function. Values of ψ are in units of m^2/s.

We can either look up the solution for cubic equations or use Newton's iteration method to obtain y for a given value of ψ. In general there are three roots – we choose the positive real root with $0 < y < h$, which is the only one that has physical meaning for this problem. Along the top wall, $y = h$, and Eq. 1 yields

Stream function along top wall:

$$\psi_{top} = -\frac{1}{12\mu}\frac{dP}{dx}h^3 = \frac{1}{12\left(1.002\times10^{-3}\text{ kg/m}\cdot\text{s}\right)}\left(20{,}000\text{ N/m}^3\right)\left(0.00120\text{ m}\right)^3\left(\frac{\text{kg}\cdot\text{m}}{\text{s}^2\text{N}}\right)$$
$$= 2.874\times10^{-3}\text{ m}^2\text{/s}$$

The streamlines themselves are straight, flat horizontal lines as seen by Eq. 1. We divide ψ_{top} by 10 to generate evenly spaced stream functions. We plot 11 streamlines in Fig. 1 (counting the streamlines on both walls) by plugging these values of ψ into Eq. 2 and solving for y.

The streamlines are not evenly spaced. This is because the volume flow rate per unit width between two streamlines ψ_2 and ψ_1 is equal to $\psi_2 - \psi_1$. The flow speeds in the middle of the channel are higher than those near the top or bottom of the channel, so we expect the streamlines to be closer near the middle.

Discussion The extent of the x axis in Fig. 1 is arbitrary since the flow is fully developed. You can immediately see from a streamline plot like Fig. 1 where flow speeds are high and low (relatively speaking).

9-52

Solution We are to calculate the volume flow rate and average speed of air being sucked through a sampling probe.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional.

Analysis For 2-D incompressible flow the difference in the value of the stream function between two streamlines is equal to the volume flow rate per unit width between the two streamlines. Thus,

Volume flow rate through the sampling probe:

$$\dot{V} = \psi_u - \psi_l \times W = (0.150 - 0.093)\ \text{m}^2/\text{s} \times 0.0395\ \text{m} = 0.0022515\ \text{m}^3/\text{s} \cong \mathbf{0.00225\ m^3/s} \qquad (1)$$

The average speed of air in the probe is obtained by dividing volume flow rate by cross-sectional area,

Average speed through the sampling probe:

$$V_{\text{avg}} = \frac{\dot{V}}{hW} = \frac{0.022515\ \text{m}^3/\text{s}}{(0.00458\ \text{m})(0.0395\ \text{m})} = 12.4454\ \text{m/s} \cong \mathbf{12.4\ m/s} \qquad (2)$$

Discussion Notice that the streamlines inside the probe are more closely packed than are those outside the probe because the flow speed is higher inside the probe.

9-53

Solution We are to sketch streamlines for the case of a sampling probe with too little suction, and we are to name this type of sampling and label the lower and upper dividing streamlines.

Analysis If the suction were too weak, the volume flow rate through the probe would be too low and the average air speed through the probe would be lower than that of the air stream. The dividing streamlines would diverge outward rather than inward as sketched in Fig. 1. We would call this type of sampling **subisokinetic sampling**.

Discussion We have drawn the streamlines inside the probe further apart than those in the air stream because the flow speed is lower inside the probe.

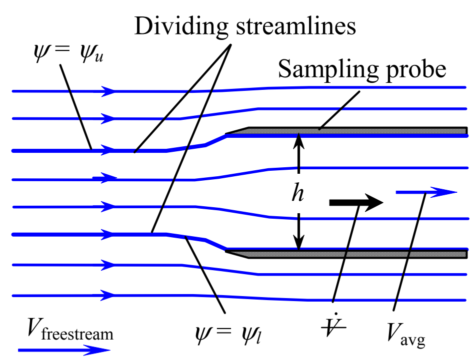


FIGURE 1
Streamlines for *subisokinetic sampling*.

9-54

Solution We are to calculate the speed of the air stream of a previous problem.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional.

Analysis In the air stream far upstream of the probe,

Volume flow rate per unit width:
$$\frac{\dot{V}}{W} = \psi_u - \psi_l = V_{\text{freestream}} y_u - V_\infty y_l = V_{\text{freestream}} \left(y_u - y_l \right) \qquad (1)$$

By definition of streamlines, the volume flow rate between the two dividing streamlines must be the same as that through the probe itself. We know the volume flow rate through the probe from the results of the previous problem. The value of the stream function on the lower and upper dividing streamlines are the same as those of the previous problem, namely ψ_l = 0.093 m^2/s and ψ_u = 0.150 m^2/s respectively. We also know $y_u - y_l$ from the information given here. Thus, Eq. 1 yields

Freestream speed:
$$V_{\text{free stream}} = \frac{\psi_u - \psi_i}{y_u - y_i} = \frac{(0.150 - 0.093)\,\text{m}^2/\text{s}}{0.00624\,\text{m}} = 9.134624\,\text{m/s} \cong \mathbf{9.13\,m/s} \qquad (2)$$

Discussion We verify by these calculations that the sampling is superisokinetic (average speed through the probe is higher than that of the upstream air stream).

9-55

Solution For a given stream function we are to generate expressions for the velocity components.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the r-θ plane.

Analysis We differentiate ψ to find the velocity components in cylindrical coordinates,

Radial velocity component:
$$\boxed{u_\theta = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V \cos\theta \left(1 - \frac{a^2}{r^2} \right)}$$

Tangential velocity component:
$$\boxed{u_\theta = -\frac{\partial \psi}{\partial r} = -V \sin\theta \left(1 + \frac{a^2}{r^2} \right)}$$

Discussion The radial velocity component is zero at the cylinder surface ($r = a$), but the tangential velocity component is not. In other words, this approximation does not satisfy the no-slip boundary condition along the cylinder surface. See Chap. 10 for a more detailed discussion about such approximations.

9-56

Solution We are to verify that the given ψ satisfies the continuity equation, and we are to discuss any restrictions.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is axisymmetric (ψ is a function of r and z only).

Analysis We plug the given velocity components into the axisymmetric continuity equation,

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\partial(u_z)}{\partial z}=\frac{1}{r}\frac{\partial\left(-\frac{\partial\psi}{\partial z}\right)}{\partial r}+\frac{\partial\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)}{\partial z}=\frac{1}{r}\left(-\frac{\partial^2\psi}{\partial r\partial z}+\frac{\partial^2\psi}{\partial z\partial r}\right)=0$$

Thus we see that **continuity is satisfied by the given stream function**. The only restriction on ψ is that $\boldsymbol{\psi}$ **must be a smooth function of *r* and *z***.

Discussion For a smooth function of two variables, the order of differentiation does not matter.

9-57

Solution For a given velocity field we are to generate an expression for ψ.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis We start by picking one of the two definitions of the stream function (it doesn't matter which part we choose – the solution will be identical).

$$\frac{\partial\psi}{\partial y}=u=V\cos\alpha \qquad (1)$$

Next we integrate Eq. 1 with respect to y, noting that this is a *partial* integration and we must add an arbitrary function of the other variable, x, rather than a simple constant of integration.

$$\psi=yV\cos\alpha+g(x) \qquad (2)$$

Now we choose the other part of the definition of ψ, differentiate Eq. 2, and rearrange as follows:

$$v=-\frac{\partial\psi}{\partial x}=-g'(x) \qquad (3)$$

where $g'(x)$ denotes dg/dx since g is a function of only one variable, x. We now have two expressions for velocity component v, the given equation and Eq. 3. We equate these and integrate with respect to x to find $g(x)$,

$$v=V\sin\alpha=-g'(x) \qquad g'(x)=-V\sin\alpha \qquad g(x)=-xV\sin\alpha+C \qquad (4)$$

Note that here we have added an arbitrary constant of integration C since g is a function of x only. Finally, plugging Eq. 4 into Eq. 2 yields the final expression for ψ,

Stream function: $$\boxed{\psi=V(y\cos\alpha-x\sin\alpha)+C} \qquad (5)$$

Constant C is arbitrary; it is common to set it to zero, although it can be set to any desired value.

Discussion You can verify by differentiating ψ that Eq. 5 yields the correct values of u and v.

9-58

Solution For a given stream function, we are to calculate the velocity components and verify incompressibility.

Assumptions **1** The flow is steady. **2** The flow is incompressible (this assumption is to be verified). **3** The flow is two-dimensional in the *x-y* plane, implying that $w = 0$ and neither u nor v depend on z.

Analysis (*a*) We use the definition of ψ to obtain expressions for u and v.

Velocity components:
$$\boxed{u = \frac{\partial \psi}{\partial y} = bx + 2cy \qquad v = -\frac{\partial \psi}{\partial x} = -2ax - by} \qquad (1)$$

(*b*) We check if the incompressible continuity equation in the *x-y* plane is satisfied by the velocity components of Eq. 1,

Incompressible continuity:
$$\underbrace{\frac{\partial u}{\partial x}}_{b} + \underbrace{\frac{\partial v}{\partial y}}_{-b} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0} = 0 \qquad b - b = 0 \qquad (2)$$

We conclude that the flow is indeed incompressible.

Discussion Since ψ is a smooth function of x and y, it automatically satisfies the continuity equation by its definition. Equation 2 confirms this. If it did not, we would go back and look for an algebra mistake somewhere.

9-59

Solution We are to plot several streamlines for a given velocity field.

Analysis We re-write the stream function equation of the previous problem with all the terms on one side,

$$cy^2 + bxy + ax^2 - \psi = 0 \qquad (1)$$

For any constant value of ψ (along a streamline), Eq. 1 is in a form that enables us to use the quadratic rule to solve for y as a function of x,

Equation for a streamline:
$$y = \frac{-bx \pm \sqrt{b^2x^2 - 4c\left(ax^2 - \psi\right)}}{2c} \qquad (2)$$

We plot the streamlines in Fig. 1. For each value of ψ there are two curves – one for the positive root and one for the negative root of Eq. 2. There is symmetry about a diagonal line through the origin. The streamlines appear to be hyperbolae. We determine the flow direction by plugging in a couple values of x and y and calculating the velocity components; e.g., at $x = 1$ m and $y = 3$ m, $u = 2.7$ m/s and $v = 4.9$ m/s. The flow at this point is in the upper right direction. Similarly, at $x = 1$ m and $y = -2$ m, $u = -3.3$ m/s and $v = -1.6$ m/s. The flow at this point is in the lower left direction.

Discussion This flow may not represent any particular physical flow field, but it produces an interesting flow pattern.

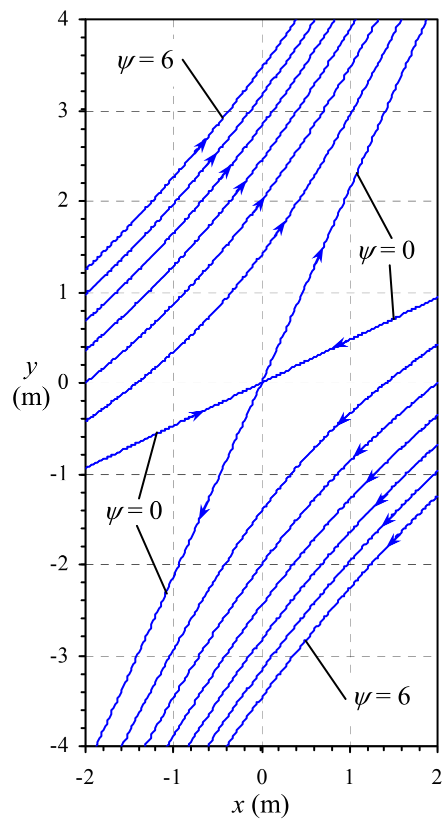


FIGURE 1
Streamlines for a given velocity field. Values of ψ are in units of m^2/s.

9-60

Solution For a given stream function, we are to calculate the velocity components and verify incompressibility.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane, implying that $w = 0$ and neither u nor v depend on z.

Analysis (*a*) We use the definition of ψ to obtain expressions for u and v.

Velocity components:
$$\boxed{u = \frac{\partial \psi}{\partial y} = -2by + dx \qquad v = -\frac{\partial \psi}{\partial x} = -2ax - c - dy} \tag{1}$$

(*b*) We check if the incompressible continuity equation in the *x*-*y* plane is satisfied by the velocity components of Eq. 1,

Incompressible continuity:
$$\underbrace{\frac{\partial u}{\partial x}}_{d} + \underbrace{\frac{\partial v}{\partial y}}_{-d} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0} = 0 \qquad d - d = 0 \tag{2}$$

We conclude that the flow is indeed incompressible.

Discussion Since ψ is a smooth function of x and y, it automatically satisfies the continuity equation by its definition. Eq. 2 confirms this. If it did not, we would go back and look for an algebra mistake somewhere.

9-61

Solution We are to make up a stream function $\psi(x,y)$, calculate the velocity components and verify incompressibility.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis Every student should have a different stream function. He or she then takes the derivatives with respect to y and x to find u and v. The student should then plug his/her velocity components into the incompressible continuity equation. **Continuity will be satisfied regardless of $\psi(x,y)$, provided that $\psi(x,y)$ is a smooth function of x and y**.

Discussion As long as ψ is a smooth function of x and y, it automatically satisfies the continuity equation by its definition.

9-62

Solution We are to calculate the percentage of flow going through one branch of a branching duct.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis For 2-D incompressible flow the difference in the value of the stream function between two streamlines is equal to the volume flow rate per unit width between the two streamlines. Thus,

Main branch:
$$\left.\frac{\dot{V}}{W}\right)_{\text{main}} = \psi_{\text{upper wall}} - \psi_{\text{lower wall}} = \left(4.35 - 2.03\right)\frac{\text{m}^2}{\text{s}} = 2.32\frac{\text{m}^2}{\text{s}} \qquad (1)$$

Similary, in the upper branch,

Upper branch:
$$\left.\frac{\dot{V}}{W}\right)_{\text{upper}} = \psi_{\text{upper wall}} - \psi_{\text{branch wall}} = (4.35 - 3.10)\frac{\text{m}^2}{\text{s}} = 1.25\frac{\text{m}^2}{\text{s}} \qquad (2)$$

On a percentage basis, the percentage of volume flow through the upper branch is calculated as

$$\frac{\dot{V}_{\text{upper}}}{\dot{V}_{\text{main}}} = \frac{\left.\frac{\dot{V}}{W}\right)_{\text{upper}}}{\left.\frac{\dot{V}}{W}\right)_{\text{main}}} = \frac{1.25\frac{\text{m}^2}{\text{s}}}{2.32\frac{\text{m}^2}{\text{s}}} = 0.53879 \cong \mathbf{53.9\%} \qquad (3)$$

Discussion No dimensions are given, so it is not possible to calculate velocities.

9-63

Solution We are to calculate duct height *h* for a given average velocity through a duct and values of stream function along the duct walls.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis The volume flow rate through the main branch of the duct is equal to the average velocity times the cross-sectional area of the duct,

Volume flow rate:
$$\dot{V} = V_{\text{avg}} W h \tag{1}$$

We solve for h in Eq. 1, using the results of the previous problem,

Duct height:
$$h = \frac{1}{V_{\text{avg}}} \frac{\dot{V}}{W}\bigg)_{\text{main}} = \frac{1}{13.4 \frac{\text{m}}{\text{s}}} \times 2.32 \frac{\text{m}^2}{\text{s}} \left(\frac{100 \text{ cm}}{\text{m}} \right) = 17.3134 \text{ cm} \cong \mathbf{17.3\ cm} \tag{2}$$

An alternative way to solve for height *h* is to assume uniform flow in the main branch, for which $\psi = V_{\text{avg}} y$. We take the difference between ψ at the top of the duct and ψ at the bottom of the duct to find *h*,

$$\psi_{\text{upper wall}} - \psi_{\text{lower wall}} = V_{\text{avg}} y_{\text{upper wall}} - V_{\text{avg}} y_{\text{lower wall}} = V_{\text{avg}} \left(y_{\text{upper wall}} - y_{\text{lower wall}} \right) = V_{\text{avg}} h$$

Thus,

Duct height:
$$h = \frac{\psi_{\text{upper wall}} - \psi_{\text{lower wall}}}{V_{\text{avg}}} = \frac{(4.35 - 2.03) \frac{\text{m}^2}{\text{s}}}{13.4 \frac{\text{m}}{\text{s}}} \left(\frac{100 \text{ cm}}{\text{m}} \right) = 17.3134 \text{ cm} \cong \mathbf{17.3\ cm} \tag{3}$$

You can see that we get the same result as that of Eq. 2.

Discussion The result is correct even if the velocity profile through the duct is not uniform, since we have used the *average* velocity in our calculations.

9-64E

Solution For a given velocity field we are to generate an expression for ψ and plot several streamlines for given values of constants a and b.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y plane, implying that $w = 0$ and neither u nor v depend on z.

Analysis We plug the given equation into the steady incompressible continuity equation,

Condition for incompressibility: $$\frac{\partial v}{\partial y} = -\underbrace{\frac{\partial u}{\partial x}}_{2ax-by} - \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0} \qquad \frac{\partial v}{\partial y} = -2ax + by$$

Next we integrate with respect to y. Note that since the integration is a *partial* integration, we must add some arbitrary function of x instead of simply a constant of integration.

y component of velocity: $$v = -2axy + \frac{by^2}{2} + f(x)$$

If the flow were three-dimensional, we would add a function of x and z instead. We are told that $v = 0$ for all values of x when $y = 0$. This is only possible if $f(x) = 0$. Thus,

y component of velocity: $$v = -2axy + \frac{by^2}{2} \qquad (1)$$

To obtain the stream function, we start by picking one of the two parts of the definition of the stream function,

$$\frac{\partial \psi}{\partial y} = u = ax^2 - bxy$$

Next we integrate the above equation with respect to y, noting that this is a *partial* integration and we must add an arbitrary function of the other variable, x, rather than a simple constant of integration.

$$\psi = ax^2 y - \frac{bxy^2}{2} + g(x) \qquad (2)$$

Now we choose the other part of the definition of ψ, differentiate Eq. 2, and rearrange as follows:

$$v = -\frac{\partial \psi}{\partial x} = -2axy + \frac{by^2}{2} - g'(x) \qquad (3)$$

where $g'(x)$ denotes dg/dx since g is a function of only one variable, x. We now have two expressions for velocity component v, Eq. 1 and Eq. 3. We equate these and integrate with respect to x to find $g(x)$,

$$g'(x) = 0 \qquad g(x) = C \qquad (4)$$

Note that here we have added an arbitrary constant of integration C since g is a function of x only. But C must be zero in order for ψ to be zero for any value of x when $y = 0$. Finally, Eq. 2 yields the final expression for ψ,

Solution: $$\boxed{\psi = ax^2 y - \frac{bxy^2}{2}} \qquad (5)$$

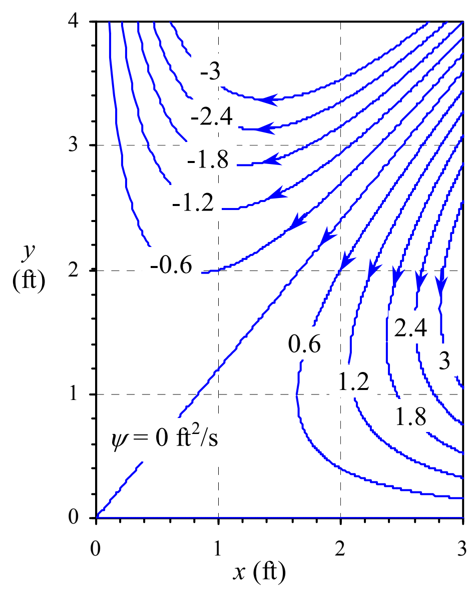


FIGURE 1
Streamlines for a given velocity field; the value of constant ψ is indicated for each streamline in units of ft^2/s.

To plot the streamlines, we note that Eq. 5 represents a *family* of curves, one unique curve for each value of the stream function ψ. We solve Eq. 5 for y as a function of x. A bit of algebra (the quadratic rule) yields

Equation for streamlines:
$$y = \frac{ax^2 \pm \sqrt{a^2x^4 - 2\psi bx}}{bx} \qquad (6)$$

For the given values of constants a and b, we plot Eq. 6 for several values of ψ in Fig. 1; these curves of constant ψ are streamlines of the flow. Note that both the positive and negative roots of Eq. 6 are required to plot each streamline. The direction of the flow is found by calculating u and v at some point in the flow field. We pick x = 2 ft, y = 2 ft, where u = −1.2 ft/s and v = −2.1 ft/s. This indicates flow to the lower left near this location. We fill in the rest of the arrows in Fig. 1 to be consistent. We see that the flow enters from the upper right, and splits into two parts – one to the lower right and one to the upper left.

Discussion It is always a good idea to check your algebra. In this example, you should differentiate Eq. 5 to verify that the velocity components of the given equation are obtained.

9-65

Solution We are to generate an expression for the stream function that describes a given velocity field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is axisymmetric (ψ is a function of r and z only).

Analysis The r and z velocity components from Problem 9-34 are

Velocity field:
$$u_r = -\frac{r}{2}\frac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L} \qquad u_z = u_{z,\text{entrance}} + \frac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L}z \qquad (1)$$

To generate the stream function we use the definition of ψ for steady, incompressible, axisymmetric flow,

Axisymmetric stream function:
$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \qquad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \qquad (2)$$

We choose one of the definitions of Eq. 2 to integrate. We pick the second one,

Integration:
$$\begin{aligned}\psi &= \int r u_z dr = \int r\left(u_{z,\text{entrance}} + \frac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L}z\right)dr \\ &= \frac{r^2}{2}\left(u_{z,\text{entrance}} + \frac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L}z\right) + f(z)\end{aligned} \qquad (3)$$

We added a function of z instead of a constant of integration since this is a partial integration. Now we take the z derivative of Eq. 3 and use the other half of Eq. 2,

Differentiation:
$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} = -\frac{r}{2}\frac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L} - \frac{1}{r}f'(z) \qquad (4)$$

We equate Eq. 4 to the known value of u_r from Eq. 1,

Comparison:
$$u_r = -\frac{r}{2}\frac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L} - \frac{1}{r}f'(z) = -\frac{r}{2}\frac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L} \quad \text{or} \quad f'(z) = 0 \qquad (5)$$

Since f is a function of z only, integration of Eq. 5 yields $f(z)$ = constant. The final result is thus

Stream function:
$$\boxed{\psi = \frac{r^2}{2}\left(u_{z,\text{entrance}} + \frac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L}z\right) + \text{constant}} \qquad (6)$$

Discussion The constant of integration can be any value since velocity components are determined by taking *derivatives* of the stream function.

9-66E

Solution We are to calculate the axial speed at the entrance and exit of the nozzle, and we are to plot several streamlines for a given axisymmetric flow field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is axisymmetric (ψ is a function of r and z only).

Analysis (*a*) Since u_z is not a function of radius, the axial velocity profile across a cross section of the nozzle is uniform. (This is consistent with the assumption that frictional effects along the nozzle walls are neglected.) Thus, at any cross section the axial speed is equal to the volume flow rate divided by cross-sectional area,

Entrance axial speed:

$$u_{z,\text{entrance}} = \frac{4\dot{V}}{\pi D_{\text{entrance}}^{\ 2}} = \frac{4\times 2.0\ \dfrac{\text{gal}}{\text{min}}}{\pi\left(0.50\text{ in}\right)^2}\left(\frac{0.1337\text{ ft}^3}{\text{gal}}\right)\left(\frac{12\text{ in}}{\text{ft}}\right)^2\left(\frac{\text{min}}{60\text{ s}}\right) = \mathbf{3.268\ \frac{ft}{s}} \qquad (1)$$

Similarly,

Exit axial speed:

$$u_{z,\text{exit}} = \frac{4\dot{V}}{\pi D_{\text{exit}}^{\ 2}} = \mathbf{41.69\ \frac{ft}{s}} \qquad (2)$$

(*b*) We use the stream function developed in Problem 9-61. Setting the constant to zero for simplicity, we have

Stream function:

$$\psi = \frac{r^2}{2}\left(u_{z,\text{entrance}} + \frac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L}z\right) \qquad (3)$$

We solve Eq. 3 for r as a function of z and plot several streamlines in Fig. 1,

Streamlines:

$$r = \pm\sqrt{\frac{2\psi}{u_{z,\text{entrance}} + \dfrac{u_{z,\text{exit}} - u_{z,\text{entrance}}}{L}z}} \qquad (4)$$

At the nozzle entrance (z = 0), the wall is at $r = D_{\text{entrance}}/2$ = 0.25 inches. Eq. 3 yields ψ_{wall} = 0.0007073 ft^3/s for the streamline that passes through this point. This streamline thus represents the shape of the nozzle wall, and we have designed the nozzle shape.

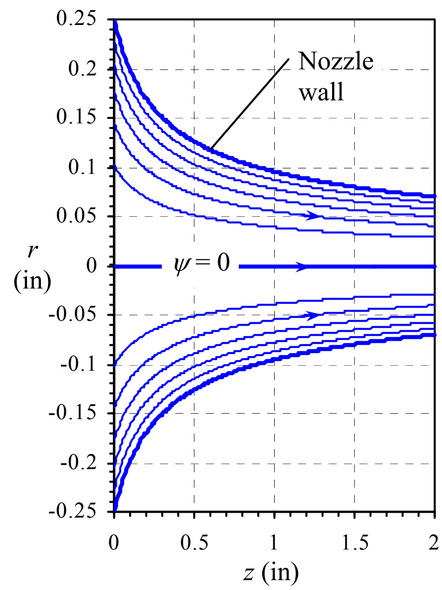


FIGURE 1
Streamlines for flow through an axisymmetric garden hose nozzle. Note that the vertical axis is highly magnified.

Discussion You can verify that the diameter between the outermost streamlines varies from D_{entrance} to D_{exit}.

9-67

Solution We are to discuss the sign of the stream function in a separation bubble, and determine where ψ is a minimum.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis For 2-D incompressible flow the difference in the value of the stream function between two streamlines is equal to the volume flow rate per unit width between the two streamlines. For example, $\psi_{upper} - \psi_0$ is positive and represents the volume flow rate per unit width between the wall and the uppermost streamline, The flow between these two streamlines is to the right. Likewise, the difference between ψ along the dividing streamline and $\psi = \psi_1$ along a streamline in the upper part of the separation bubble must also be positive as sketched in Fig. 1. The arc-shaped dividing streamline divides fluid within the separation bubble from fluid outside of the separation bubble. The stream function along this dividing streamline must be zero since it intersects the wall where $\psi = 0$. The only way we can have flow to the right in the upper part of the separation bubble is if ψ_1 is *negative* (Fig. 1). We conclude that **for this problem, all streamlines within the separation bubble have negative values of stream function**. **The minimum value of ψ occurs in the center of the separation bubble as sketched in Fig. 1**.

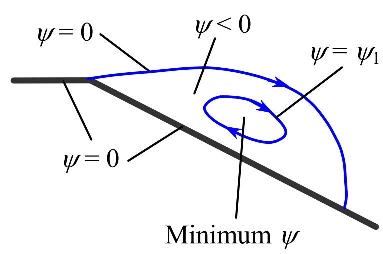


FIGURE 1
Close-up of streamlines near the separation bubble. The minimum value of the stream function occurs in the middle of the separation bubble.

Discussion We cannot conclude that ψ is always negative within a separation bubble, since we can add any arbitrary constant to all the ψ values, and it will not change the flow.

9-68

Solution We are to discuss how someone can interpret the relative speed of a flow based solely on contours of constant stream function.

Assumptions **1** The flow is steady. **2** The flow is incompressible.

Analysis For 2-D incompressible flow the difference in the value of the stream function between two streamlines is equal to the volume flow rate per unit width between the two streamlines. Thus, if the streamlines are very close together, the speed of the fluid between them is large relative to locations where the same two streamlines are far apart. **Professor Flows noticed a region in which the streamlines were very close together, implying high relative speed in that region of the flow**.

Discussion If the values of ψ on the contour plot are labeled, we can actually infer the fluid speed by measuring the distance between streamlines.

9-69

Solution For the given set of streamlines, we are to discuss how we can tell the relative speed of the fluid.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is axisymmetric.

Analysis As with 2-D flow, when streamlines that are initially equally spaced spread away from each other, it indicates that the flow speed has decreased in that region. Likewise, if the streamlines come closer together, the flow speed has increased in that region. From the figure provided in the problem statement, we infer that the flow far upstream of the plate is straight and uniform, since the streamlines are parallel. The fluid decelerates as it approaches the front face of the cylinder, especially near the stagnation point, as indicated by the wide gap between streamlines. The flow accelerates rapidly to very high speeds around the corner of the cylinder as indicated by the tightly spaced streamlines there. The flow is seen to separate on top of the cylinder. Since the streamlines are very sparse in this region, **we infer that the fluid moves relatively slowly inside the separation bubble**.

Discussion Such analyses in axisymmetric flow fields are more difficult than those in 2-D planar flow fields because streamlines of equally spaced stream function are *not* spaced equally apart in a uniform axisymmetric flow field. This is due to the fact that the cross-sectional area between streamlines increases with radius (a factor of $2\pi r$ is introduced). Nevertheless, we can still tell where the flow speeds up and slows down in this example.

9-70E

Solution We are to interpret a streamline plot by determining the direction of flow and by estimating the speed of the flow at a point.

Assumptions **1** The flow is steady. **2** The flow is incompressible **3** The flow is two-dimensional.

Analysis (a) We must tilt our heads nearly upside down to see an increase in stream function ψ in the mathematically positive manner. In other words, since ψ increases in the downward direction, the flow is to the lower left, following our left side rule. Arrows are drawn in Fig. 1.

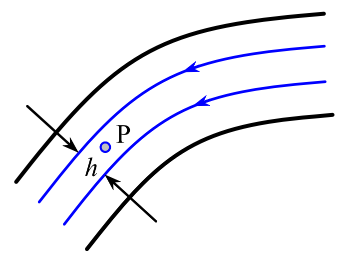


FIGURE 1
Streamlines with direction shown.

(b) For 2-D incompressible flow the difference in the value of the stream function between two streamlines is equal to the volume flow rate per unit width between the two streamlines. We approximate the flow as uniform between the two labeled streamlines in the figure provided in the problem statement. The speed at point P is thus

$$V_P \approx \frac{\dot{V}}{Wh} = \frac{1}{h}\frac{\dot{V}}{W} = \frac{1}{h}(\psi_1 - \psi_2) = \frac{1}{1.58\text{ in}}(0.45 - 0.32)\frac{\text{ft}^2}{\text{s}}\left(\frac{12\text{ in}}{\text{ft}}\right) = 0.987342\frac{\text{ft}}{\text{s}} \cong \mathbf{0.99}\frac{\mathbf{ft}}{\mathbf{s}} \quad (1)$$

(c) Nowhere did we use any property of the fluid, so changing to water does not change our result. **For either air or water (or any incompressible fluid), V_P = 0.99 ft/s** (to two significant digits).

Discussion Streamlines and stream functions are *kinematic* properties, as discussed in Chap. 4. That is why fluid density, viscosity, etc. are irrelevant here.

9-71

Solution We are to find the primary dimensions and primary units of the compressible stream function.

Analysis From the given definition, we see that ψ_ρ is the product of a density, a velocity, and a length,

Primary dimensions of ψ_ρ:
$$\boxed{\left\{\psi_\rho\right\} = \left\{\frac{\text{mass}}{\text{length}^3}\times\frac{\text{length}}{\text{time}}\times\text{length}\right\} = \left\{\frac{\text{m}}{\text{Lt}}\right\}}$$

The primary units of ψ_ρ are kg/(m·s) (SI) and lbm/(ft s) (English).

Discussion Ironically, although the stream function is often applied to potential flows where viscosity is not a parameter, ψ_ρ has the same units as μ.

9-72

Solution We are to generate an expression for the compressible stream function for a given flow field.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

Analysis We start by picking one of the two definitions of the compressible stream function (it doesn't matter which part we choose – the solution will be identical).

$$\frac{\partial\psi_\rho}{\partial y} = \rho u = \left(\rho_1 + C_\rho x\right)\left(u_1 + C_u x\right) = \rho_1 u_1 + \left(\rho_1 C_u + u_1 C_\rho\right)x + C_\rho C_u x^2 \quad (1)$$

Next we integrate Eq. 1 with respect to *y*, noting that this is a *partial* integration and we must add an arbitrary function of the other variable, *x*, rather than a simple constant of integration.

$$\psi_\rho = \rho_1 u_1 y + \left(\rho_1 C_u + u_1 C_\rho\right)xy + C_\rho C_u x^2 y + g\left(x\right) \quad (2)$$

Now we choose the other part of the definition of ψ, differentiate Eq. 2, and rearrange as follows:

$$-\rho v = \frac{\partial\psi_\rho}{\partial x} = \left(\rho_1 C_u + u_1 C_\rho\right)y + 2C_\rho C_u xy + g'\left(x\right) \quad (3)$$

where $g'(x)$ denotes dg/dx since *g* is a function of only one variable, *x*. We now have two expressions for $-\rho v$, Eq. 3 and the value computed from the known density and velocity, i.e.

$$-\rho v = \left(\rho_1 C_u + u_1 C_\rho\right)y - 2C_u C_\rho xy \quad (4)$$

We equate Eqs. 3 and 4 and integrate with respect to *x* to find $g(x)$,

$$g'\left(x\right) = 0 \qquad g\left(x\right) = C \quad (5)$$

Note that here we have added an arbitrary constant of integration *C* since *g* is a function of *x* only. Plugging Eq. 5 into Eq. 2 yields

$$\psi_\rho = \rho_1 u_1 y + \left(\rho_1 C_u + u_1 C_\rho\right)xy + C_\rho C_u x^2 y + C \quad (6)$$

We determine constant *C* by setting $\psi_\rho = 0$ at $y = 0$ in Eq. 6, yielding $C = 0$. Thus the final expression for the compressible stream function is

Compressible stream function:
$$\boxed{\psi_\rho = \rho_1 u_1 y + \left(\rho_1 C_u + u_1 C_\rho\right)xy + C_\rho C_u x^2 y} \quad (7)$$

Discussion You can verify by differentiating ψ_ρ that Eq. 7 yields the correct values of *u* and *v*.

9-73

Solution We are to generate an expression for the compressible stream function for a given flow field.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

Analysis We start by picking one of the two definitions of the compressible stream function (it doesn't matter which part we choose – the solution will be identical).

$$\frac{\partial \psi_\rho}{\partial y} = \rho u = (\rho_1 + C_\rho x)(u_1 + C_u x) = \rho_1 u_1 + (\rho_1 C_u + u_1 C_\rho)x + C_\rho C_u x^2 \qquad (1)$$

Next we integrate Eq. 1 with respect to y, noting that this is a *partial* integration and we must add an arbitrary function of the other variable, x, rather than a simple constant of integration.

$$\psi_\rho = \rho_1 u_1 y + (\rho_1 C_u + u_1 C_\rho)xy + C_\rho C_u x^2 y + g(x) \qquad (2)$$

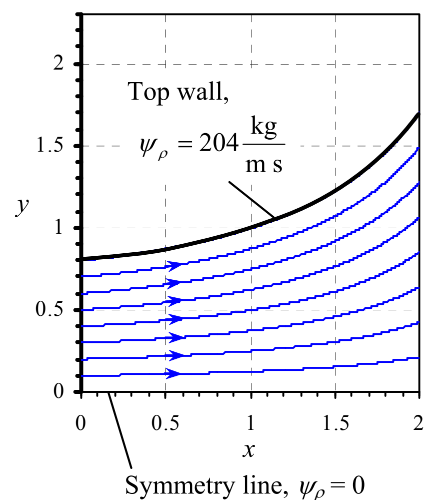


FIGURE 1
Streamlines for a diverging duct.

Now we choose the other part of the definition of ψ, differentiate Eq. 2, and rearrange as follows:

$$-\rho v = \frac{\partial \psi_\rho}{\partial x} = (\rho_1 C_u + u_1 C_\rho)y + 2C_\rho C_u xy + g'(x) \qquad (3)$$

where $g'(x)$ denotes dg/dx since g is a function of only one variable, x. We now have two expressions for $-\rho v$, Eq. 3 and the value computed from the known density and velocity, i.e.

$$-\rho v = (\rho_1 C_u + u_1 C_\rho)y - 2C_u C_\rho xy \qquad (4)$$

We equate Eqs. 3 and 4 and integrate with respect to x to find $g(x)$,

$$g'(x) = 0 \qquad g(x) = C \qquad (5)$$

Note that here we have added an arbitrary constant of integration C since g is a function of x only. Plugging Eq. 5 into Eq. 2 yields

$$\psi_\rho = \rho_1 u_1 y + (\rho_1 C_u + u_1 C_\rho)xy + C_\rho C_u x^2 y + C \qquad (6)$$

We determine constant C by setting $\psi_\rho = 0$ at $y = 0$ in Eq. 6, yielding $C = 0$. Thus the final expression for the compressible stream function is

Compressible stream function: $$\boxed{\psi_\rho = \rho_1 u_1 y + (\rho_1 C_u + u_1 C_\rho)xy + C_\rho C_u x^2 y} \qquad (7)$$

We solve Eq. 7 for y as a function of x and ψ_ρ so that we can plot streamlines,

Equation for plotting streamlines: $$y = \frac{\psi_\rho}{\rho_1 u_1 + (\rho_1 C_u + u_1 C_\rho)x + C_\rho C_u x^2} \qquad (8)$$

We plot Eq. 8 in Fig. 1 for several values of ψ_ρ, using the values of constants u_1, ρ_1, C_u, and C_ρ given in Problem 9-21. The agreement with the streamlines of Problem 9-21 is excellent.

The streamline starting at $x = 0$, $y = 0.8$ m is the top wall of the duct. Therefore the value of ψ_ρ at the top wall of the diverging duct is found be setting at $x = 0$ and $y = 0.8$ m,

ψ_ρ at the top wall: $$\psi_{\rho,\text{top}} = \rho_1 u_1 y = \left(0.85 \frac{\text{kg}}{\text{m}^3}\right)\left(300 \frac{\text{m}}{\text{s}}\right)(0.8 \text{ m}) = \mathbf{204 \frac{kg}{m \cdot s}} \qquad (9)$$

Discussion You can verify by differentiating ψ_ρ that Eq. 9 yields the correct values of u and v.

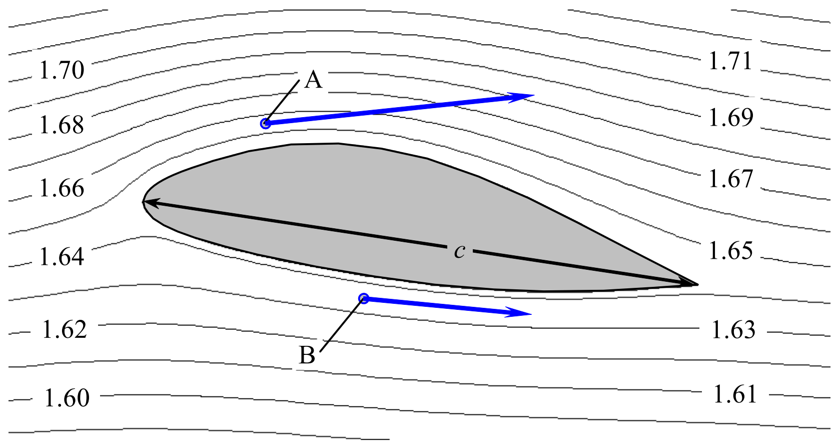
9-74

Solution We are to interpret a streamline plot by determining the direction of flow and by estimating the speed of the flow at a point.

Assumptions **1** The flow is steady. **2** The flow is incompressible **3** The flow is two-dimensional.

Analysis (*a*) We can tell the direction of the flow by whether ψ_ρ increases or decreases in the vertical direction (left side rule). We see that at points A and B the flow is to the right. Furthermore, since the streamlines near point B are somewhat further apart than those near point A (by a factor of about 1.6), the speed at point A is a factor of about 1.6 greater than that at point B. Arrows are drawn in Fig. 1.

FIGURE 1
Relative velocity vectors at points A and B, added to the streamline plot.



In terms of lift, it is obvious that the flow speeds near the upper surface of the hydrofoil are greater than those near the lower surface. From the Bernoulli equation we know that low speeds lead to (relatively) higher pressures; thus the pressure on the lower half of the hydrofoil is greater than that on the upper half, leading to lift.

(*b*) For 2-D incompressible flow the difference in the value of the stream function between two streamlines is equal to the volume flow rate per unit width between the two streamlines. We approximate the flow as uniform between the two streamlines that enclose point A in Fig. P9-70. By measurement with a ruler, we find that the distance δ between streamlines 1.65 and 1.66 is about 0.034c, or about (0.034)(9.0 mm) = 0.306 mm. The speed at point A is thus

$$V_A \approx \frac{\dot{V}}{W\delta} = \frac{1}{\delta}\frac{\dot{V}}{W} = \frac{1}{\delta}\left(\psi_{1.66} - \psi_{1.65}\right)$$
$$= \frac{1}{0.306\times10^{-3}\text{ m}}\left(1.66 - 1.65\right)\frac{\text{m}^2}{\text{s}} = 32.7\frac{\text{m}}{\text{s}} \cong \mathbf{33.\frac{m}{s}}$$

We give our answer to only two significant digits here because of the difficulty of measuring the distance between the two streamlines.

Discussion Students' answers may vary somewhat depending on how accurately they measure the distance between streamlines. Values between 30 and 40 m/s are reasonable.

9-75

Solution We are to interpret a streamline plot by determining the direction of flow and by estimating the speed of the flow at a point.

Assumptions **1** The flow is steady (time-averaged). **2** The flow is incompressible **3** The flow is two-dimensional.

Properties The density of air at $T = 20^{\circ}\text{C}$ is 1.18 kg/m^3.

Analysis (*a*) We can tell the direction of the flow by whether ψ_ρ increases or decreases in the vertical direction (left side rule). We see that at point A, the flow is to the left, while at point B the flow is to the right. Furthermore, since the streamlines near point B are much closer together than those near point A (by a factor of about five), the speed at point B is a factor of about five greater than that at point A. Arrows are drawn in Fig. 1.

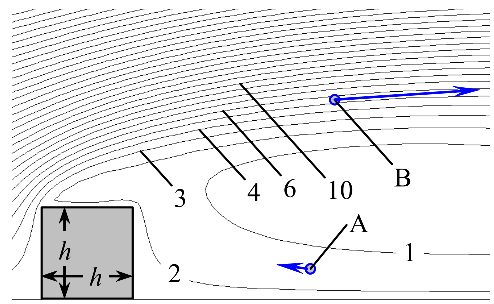


FIGURE 1
Relative velocity vectors at points A and B, added to the streamline plot.

(*b*) For 2-D incompressible flow the difference in the value of the compressible stream function between two streamlines is equal to the mass flow rate per unit width between the two streamlines. We approximate the flow as uniform between the two streamlines that enclose point B in Fig. P9-71. By measurement with a ruler, we find that the distance δ between streamlines 5 and 6 is about h/10, or about 0.10 m. The speed at point B is thus

$$V_{\text{B}} \approx \frac{\dot{m}}{\rho W \delta} = \frac{1}{\rho\delta}\frac{\dot{m}}{W} = \frac{1}{\rho\delta}\left(\psi_6 - \psi_5\right) = \frac{1}{\left(1.18\ \frac{\text{kg}}{\text{m}^3}\right)\left(0.10\ \text{m}\right)}\left(6-5\right)\frac{\text{kg}}{\text{m}\cdot\text{s}} = \mathbf{8.5\ \frac{m}{s}} \qquad (1)$$

We are only accurate to one digit here because of the difficulty of measuring the distance between the two streamlines. We give our final result as $\boldsymbol{V_{\text{B}}}$ **= 8 or 9 m/s**.

Discussion Students' answers may vary considerably depending on how accurately they measure the distance between streamlines.

9-76

Solution We are to determine the value of the stream function along the positive *y* axis and the negative *x* axis for the case of a line source at the origin.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* or r-θ plane.

Analysis For 2-D incompressible flow the difference in the value of the stream function between two streamlines is equal to the volume flow rate per unit width between the two streamlines. Let us take the arc of the circle of radius *r* between the positive *x* axis and the positive *y* axis of the figure provided with the problem statement. The volume flow rate per unit width through this arc is one-fourth of $\dot{V}/L$, the total volume flow rate per unit width, since the arc spans exactly one-fourth of the circumference of the circle.

$$\psi_{\text{positive } y \text{ axis}} - \psi_{\text{positive } x \text{ axis}} = \frac{1}{4}\frac{\dot{V}}{L} \qquad (1)$$

Since $\psi = 0$ along the positive *x* axis, we conclude that

ψ along positive y axis:

$$\boxed{\psi_{\text{positive } y \text{ axis}} = \frac{1}{4}\frac{\dot{V}}{L}} \qquad (2)$$

Similarly, the volume flow rate through the top half of the circle is half of the total volume flow rate and we conclude that

ψ along negative x axis:

$$\boxed{\psi_{\text{negative } y \text{ axis}} = \frac{1}{2}\frac{\dot{V}}{L}} \qquad (3)$$

Discussion Some CFD codes use ψ as a variable, and we thus need to specify the value of ψ along boundaries of the computational domain. Simple calculations such as this are useful in these situations.

9-77

Solution We are to determine the value of the stream function along the positive *y* axis and the negative *x* axis for the case of a line sink at the origin.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* or r-θ plane.

Analysis Everything is the same as in Problem 9-59 except that the flow direction is reversed everywhere. The volume flow rate per unit width through the arc of radius *r* between the positive *x* axis and the positive *y* axis of Fig. P9-59 is now *negative* one-fourth of $\dot{V}/L$ since the flow is now mathematically negative.

$$\psi_{\text{positive } y \text{ axis}} - \psi_{\text{positive } x \text{ axis}} = -\frac{1}{4}\frac{\dot{V}}{L} \qquad (1)$$

Since $\psi = 0$ along the positive *x* axis, we conclude that

ψ along positive y axis:

$$\boxed{\psi_{\text{positive } y \text{ axis}} = -\frac{1}{4}\frac{\dot{V}}{L}} \qquad (2)$$

Similarly, the volume flow rate through the top half of the circle is half of the total volume flow rate and is negative. We conclude that

ψ along negative x axis:

$$\boxed{\psi_{\text{negative } x \text{ axis}} = -\frac{1}{2}\frac{\dot{V}}{L}} \qquad (3)$$

Discussion We need to be careful of the sign of ψ.

Linear Momentum Equation, Boundary Conditions, and Applications

9-78C

Solution We are to define mechanical pressure and discuss its application.

Analysis **Mechanical pressure is the mean normal stress acting inwardly on a fluid element**. For an incompressible fluid, the density is constant and therefore we have no equation of state available for calculation of the thermodynamic pressure. In fact, thermodynamic pressure cannot even be defined for an incompressible fluid. Fluid elements and surfaces still "feel" a pressure, however, and this pressure is the so-called mechanical pressure.

Discussion When dealing with incompressible fluid flows, pressure variable P is always interpreted as the mechanical pressure P_m.

9-79C

Solution We are to describe the constitutive equations and name the equation to which they are applied.

Analysis **The constitutive equations are relationships between the components of the stress tensor and the primary unknowns of the problem, namely pressure and velocity**. The constitutive equations enable us to write the components of the stress tensor in *Cauchy's equation* in terms of the velocity field and the pressure field.

Discussion Cauchy's equation by itself is useless without the constitutive equations, because we would have too many unknowns for the number of available equations.

9-80C

Solution We are to discuss velocity boundary conditions in a stationary and a moving frame of reference for the case of an airplane flying through the air.

Analysis (*a*) From the stationary frame of reference, $\vec{V} = \vec{V}_{\text{airplane}}$ on all surfaces of the airplane, (no-slip boundary condition). Far from the airplane **the air is still** ($\vec{V}$ **= 0**).

(*b*) From the reference frame moving with the airplane, $\vec{V} = 0$ on all surfaces, (no-slip boundary condition). Far from the airplane the air is moving towards the airplane at **a speed that is opposite the airplane's speed** ($\vec{V} = -\vec{V}_{\text{airplane}}$).

Discussion The no-slip condition requires that the fluid velocity equal the airplane velocity everywhere on the airplane surface, regardless of the geometry of the airplane, and regardless of the frame of reference.

9-81C

Solution We are to discuss the difference between Newtonian fluids and non-Newtonian fluids, and we are to give examples of each.

Analysis The main distinction between a Newtonian fluid and a non-Newtonian fluid is that **for flow of a Newtonian fluid, shear stress is linearly proportional to shear strain rate**, whereas for flow of a non-Newtonian fluid, the relationship between shear stress and shear strain rate is nonlinear.

There are many examples of Newtonian fluids. Most pure, common liquids like water, oil, gasoline, alcohol, etc. are Newtonian. Most gases also behave like Newtonian fluids. Non-newtonian fluids include paint, pastes and creams, polymer solutions, cake batter, slurries and colloidal suspensions like quicksand, blood, etc.

Discussion The Navier-Stokes equations apply only to Newtonian fluids. For non-Newtonian fluids, you would need to insert nonlinear constitutive equations into Cauchy's equations in order to obtain a useful differential equation for conservation of linear momentum.

9-82C

Solution We are to define or describe each type of fluid.

Analysis

(*a*) **A viscoelastic fluid is a fluid that returns (either fully or partially) to its original shape after the applied stress is released**.

(*b*) **A pseudoplastic fluid is a shear thinning fluid – the more the fluid is sheared, the less viscous it becomes**.

(*c*) **A dilatant fluid is a shear thickening fluid – the more the fluid is sheared, the more viscous it becomes**.

(*d*) **A Bingham plastic fluid is an extreme type of pseudoplastic fluid that requires a finite stress called the *yield stress* in order for the fluid to flow at all**.

Discussion All of the above are examples of non-Newtonian fluids.

9-83C

Solution We are to discuss each term, and write the equation as a word equation.

Analysis **Term I is the net body force acting on the control volume. Term II is the net surface force acting on the control volume. Term III is the net rate of change of linear momentum within the control volume. Term IV is the net rate of outflow of linear momentum through the control surface**. In words, the equation can be expressed as: "**The total force acting on the control volume is the sum of body forces and surface forces, and is equal to the rate at which momentum changes within the control volume plus the rate at which momentum flows out of the control volume**."

Discussion The dimensions of each term in the equation are those of momentum per time. Each term has primary dimensions of $\{mLt^{-2}\}$.

9-84

Solution We are to generate and discuss velocity and pressure boundary conditions for the given flow problem.

Assumptions **1** The flow is steady. **2** Surface tension effects are negligibly small.

Analysis We must satisfy the no-slip boundary condition on all tank walls, $\vec{V}_{\text{liquid}} = \vec{V}_{\text{tank}}$. Mathematically, we write $\boldsymbol{u_r} = \boldsymbol{u_z} = \mathbf{0}$ **and** $\boldsymbol{u_\theta} = \boldsymbol{R\omega}$ **at** $\boldsymbol{r} = \boldsymbol{R}$ (the tank side walls). We also write $\boldsymbol{u_r} = \boldsymbol{u_z} = \mathbf{0}$ **and** $\boldsymbol{u_\theta} = \boldsymbol{r\omega}$ **at z = 0** (the bottom wall of the tank). We do not specify the pressure along the tank walls. **At the free surface** $\boldsymbol{P} = \boldsymbol{P}_{\mathbf{atm}}$ since the free surface is exposed to atmospheric air. In addition, the vertical and radial components of velocity $\boldsymbol{u_z}$ **and** $\boldsymbol{u_r}$ **must equal zero at the free surface**, but the angular velocity component u_θ is set to $\boldsymbol{u_\theta} = \boldsymbol{r\omega}$ **at the free surface**. We also know that the shear stress at free surface must be zero (negligible shear due to the air). This boundary condition is not needed however since we already know the velocity field. In fact, the velocity field is known right from the start since we are told that the liquid is in solid body rotation: $u_z = u_r = 0$ and $u_\theta = r\omega$ everywhere.

Discussion This is a degenerate case of the Navier-Stokes equation since the fluid is in solid body rotation. Nevertheless, it is useful to think about the required boundary conditions.

9-85

Solution We are to compare Eqs. 1 and 2 to see if they are the same or not.

Analysis We use the product rule to differentiate Eq. 1,

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(ru_\theta \frac{-1}{r^2} + \frac{r}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right) = \mu\left(\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) + \frac{\partial u_\theta}{\partial r} \right) \qquad (3)$$

Thus we see that **Eq. 1 and Eq. 2 are equivalent**.

Discussion The viscous stress tensor is defined identically in the other texts; the terms are simply grouped together in a different fashion.

9-86

Solution We are to estimate the volume flow rate of oil between two plates, and we are to calculate the Reynolds number.

Assumptions **1** The flow is steady. **2** The oil is incompressible. **3** Since the gap is so small compared to the plate dimensions, we assume 2-D flow in the *x*-*y* plane. **4** We ignore entrance effects and end effects and assume that the flow can be approximated as fully developed channel flow everywhere in the gap.

Properties The viscosity and density of unused engine oil at T = 60°C are 72.5 × 10^{-3} kg/(m·s) and 864 kg/m^3 respectively.

Analysis The velocity field for fully developed channel flow is

Velocity components, 2-D channel flow:
$$u = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right) \qquad v = 0 \tag{1}$$

We integrate the *x* component of velocity times cross-sectional area to obtain volume flow rate (see also Problem 9-44),

Volume flow rate:
$$\dot{V} = \int_A u dA = \int_{y=0}^{y=h} \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)W dy = -\frac{1}{12\mu}\frac{dP}{dx}h^3 W \tag{2}$$

The pressure gradient is approximated as

$$\frac{dP}{dx} \approx \frac{P_{\text{out}} - P_{\text{in}}}{L} = \frac{(0-1)\text{ atm}}{1.25\text{ m}}\left(\frac{101,300\text{ N/m}^2}{\text{atm}}\right) = -81,040\text{ N/m}^3 \tag{3}$$

We plug Eq. 3 into Eq. 2 and solve for the volume flow rate,

Volume flow rate:
$$\dot{V} = -\frac{1}{12\left(72.5\times10^{-3}\ \frac{\text{kg}}{\text{m}\cdot\text{s}}\right)}\left(-81{,}040\frac{\text{N}}{\text{m}^3}\right)(0.00360\,\text{m})^3(0.550\,\text{m})\left(\frac{\text{kg m}}{\text{s}^2\text{N}}\right)$$
$$= 2.39029\times10^{-3}\ \text{m}^3/\text{s} \cong \mathbf{2.39\times10^{-3}\ m^3/s} \tag{4}$$

The average velocity of the oil through the channel is

Average velocity:
$$V = \frac{\dot{V}}{hW} = \frac{2.39029\times10^{-3}\ \text{m}^3/\text{s}}{(0.00360\,\text{m})(0.550\,\text{m})} = 1.20722\,\text{m/s} \cong 1.21\,\text{m/s} \tag{5}$$

Finally, the characteristic Reynolds number is

$$\text{Re} = \frac{\rho V h}{\mu} = \frac{(864\,\text{kg/m}^3)(1.20722\,\text{m/s})(0.00360\,\text{m})}{72.5\times10^{-3}\ \text{kg/m}\cdot\text{s}} = \mathbf{51.8} \tag{6}$$

The flow is definitely laminar.

Discussion We give our final results to three significant digits.

9-87

Solution For a given velocity field, we are to calculate the pressure field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** Gravity does not act in either the *x* or the *y* plane.

Analysis The flow field must satisfy the steady, two-dimensional, incompressible continuity *and* momentum equations. We check each equation separately; let's consider continuity first:

Continuity:
$$\underbrace{\frac{\partial u}{\partial x}}_{a}+\underbrace{\frac{\partial v}{\partial y}}_{-a}+\underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0\text{ (2-D)}}=0$$

Continuity is satisfied. Now we look at the *x component* of the Navier-Stokes equation:

x momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0\text{ (steady)}}+\underbrace{u\frac{\partial u}{\partial x}}_{(ax+b)a}+\underbrace{v\cancel{\frac{\partial u}{\partial y}}}_{(-ay+c)0}+\underbrace{\cancel{w\frac{\partial u}{\partial z}}}_{0\text{ (2-D)}}\right)=-\frac{\partial P}{\partial x}+\underbrace{\cancel{\rho g_x}}_{0}+\mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0}+\underbrace{\cancel{\frac{\partial^2 u}{\partial y^2}}}_{0}+\underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{0\text{ (2-D)}}\right) \quad (1)$$

The *x* momentum equation reduces to

x momentum:
$$\frac{\partial P}{\partial x}=\rho\left(-a^2x-ab\right) \quad (2)$$

The *x* momentum equation is satisfied provided we can generate a smooth pressure field that satisfies Eq. 2. In similar fashion (we don't show the details), the *y* momentum equation reduces to

y momentum:
$$\frac{\partial P}{\partial y}=\rho\left(-a^2y+ac\right) \quad (3)$$

The *y* momentum equation is satisfied provided we can generate a smooth pressure field that satisfies Eq. 3. The pressure field $P(x,y)$ must be a smooth function of *x* and *y*. Mathematically, this requires that the order of differentiation (*x* then *y* verses *y* then *x*) should not matter. We therefore check whether this is so by differentiating Eqs. 3 and 2 respectively:

Cross-differentiation:
$$\frac{\partial^2 P}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)=0 \qquad \frac{\partial^2 P}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right)=0 \quad (4)$$

Equation 4 shows that indeed, *P* is a smooth function of *x* and *y*. Thus, *we should be able to calculate the pressure field*. To calculate $P(x,y)$, we start with either Eq. 2 or Eq. 3 and integrate. We pick Eq. 2, which we can partially integrate (with respect to *x*) to obtain an expression for $P(x,y)$,

Pressure field from x-momentum:
$$P(x,y)=\rho\left(-\frac{a^2x^2}{2}-abx\right)+g(y) \quad (5)$$

Note that we added an arbitrary function of the other variable *y* rather than a constant of integration since this is a partial integration. We then take the partial derivative of Eq. 5 with respect to *y* to obtain

$$\frac{\partial P}{\partial y}=g'(y)=\rho\left(-a^2y+ac\right) \quad (6)$$

where we have equated our result to Eq. 3 for consistency. We can now integrate Eq. 6 to obtain the function $g(y)$:

$$g(y)=\rho\left(-\frac{a^2y^2}{2}+acy\right)+C \quad (7)$$

where *C* is an arbitrary constant of integration. Finally, we plug Eq. 7 into Eq. 5 to obtain our final expression for $P(x,y)$. The result is

Solution:

$$\boxed{P(x,y)=\rho\left(-\frac{a^2x^2}{2}-\frac{a^2y^2}{2}-abx+acy\right)+C} \qquad (8)$$

Discussion For practice, you should differentiate Eq. 8 with respect to both x and y, and compare to Eqs. 2 and 3. (This also serves as a check of our algebra.) In addition, try to obtain Eq. 8 by starting with Eq. 3 rather than Eq. 2; you should get the same answer. Pressure is found to within some arbitrary constant C since the absolute magnitude of pressure is irrelevant; only pressure *gradients* are important.

9-88

Solution For a given velocity field, we are to calculate the pressure field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** Gravity does not act in either the *x* or the *y* direction.

Analysis The flow field must satisfy the steady, two-dimensional, incompressible continuity *and* momentum equations. We check each equation separately; let's consider continuity first:

Continuity:
$$\underbrace{\frac{\partial u}{\partial x}}_{-2ax}+\underbrace{\frac{\partial v}{\partial y}}_{2ax}+\underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0\ \text{(2-D)}}=0$$

Continuity is satisfied. Now we look at the *x component* of the Navier-Stokes equation:

x momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0\ \text{(steady)}}+\underbrace{u\frac{\partial u}{\partial x}}_{\left(-ax^2\right)\left(-2ax\right)}+\underbrace{\cancel{v\frac{\partial u}{\partial y}}}_{\left(2axy\right)\left(0\right)}+\underbrace{\cancel{w\frac{\partial u}{\partial z}}}_{0\ \text{(2-D)}}\right)=-\frac{\partial P}{\partial x}+\underbrace{\cancel{\rho g_x}}_{0}+\mu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{-2a}+\underbrace{\cancel{\frac{\partial^2 u}{\partial y^2}}}_{0}+\underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{0\ \text{(2-D)}}\right) \qquad (1)$$

equation 1 reduces to

x momentum:
$$\frac{\partial P}{\partial x}=-2\rho a^2x^3-2\mu a \qquad (2)$$

The *x* momentum equation is satisfied provided we can generate a pressure field that satisfies Eq. 2. In similar fashion we examine the *y* momentum equation,

y momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{0\ \text{(steady)}}+\underbrace{u\frac{\partial v}{\partial x}}_{\left(-ax^2\right)\left(2ay\right)}+\underbrace{v\frac{\partial v}{\partial y}}_{\left(2axy\right)\left(2ax\right)}+\underbrace{\cancel{w\frac{\partial v}{\partial z}}}_{0\ \text{(2-D)}}\right)=-\frac{\partial P}{\partial y}+\underbrace{\cancel{\rho g_y}}_{0}+\mu\left(\underbrace{\cancel{\frac{\partial^2 v}{\partial x^2}}}_{0}+\underbrace{\cancel{\frac{\partial^2 v}{\partial y^2}}}_{0}+\underbrace{\cancel{\frac{\partial^2 v}{\partial z^2}}}_{0\ \text{(2-D)}}\right)$$

The *y* momentum equation reduces to

y momentum:
$$\frac{\partial P}{\partial y}=-2\rho a^2x^2y \qquad (3)$$

The *y* momentum equation is satisfied provided we can generate a pressure field that satisfies Eq. 3.

In the two-dimensional flow under discussion here, the pressure field $P(x,y)$ must be a smooth function of *x* and *y*. Mathematically, this requires that the order of differentiation (*x* then *y* versus *y* then *x*) should not matter. We check whether this is so by differentiating Eqs. 3 and 2 respectively:

Cross-differentiation:
$$\frac{\partial^2 P}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right)=0 \qquad \frac{\partial^2 P}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)=-2\rho a^2x^2 \qquad (4)$$

Since the cross-derivative terms in Eq. 4 do not match, *P* is *not* a smooth function of *x* and *y*. Thus, **we are unable to calculate a steady, incompressible, two-dimensional pressure field with the given velocity field**. We cannot proceed any further.

Discussion This problem shows that even if a velocity field satisfies the continuity equation (conservation of mass), and even if we can plot streamlines for the flow field, this does not necessarily guarantee that the velocity field is physically possible. In the present case, for instance, we are unable to find a pressure field that can satisfy the steady Navier-Stokes equation.

9-89

Solution For a given velocity field, we are to calculate the pressure field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the r-θ plane. **4** Gravity does not act in either the r or the θ direction.

Analysis The flow must satisfy the steady, two-dimensional, incompressible continuity *and* momentum equations. We check each equation separately, starting with continuity,

Continuity:
$$\underbrace{\frac{1}{r}\cancel{\frac{\partial(ru_r)}{\partial r}}}_{0}+\underbrace{\frac{1}{r}\cancel{\frac{\partial(u_\theta)}{\partial\theta}}}_{0}+\underbrace{\cancel{\frac{\partial(u_z)}{\partial z}}}_{0}=0$$

Continuity is satisfied. Now we look at the θ component of the Navier-Stokes equation,

$$\rho\left(\underbrace{\cancel{\frac{\partial u_\theta}{\partial t}}}_{0\text{ (steady)}}+\underbrace{u_r\frac{\partial u_\theta}{\partial r}}_{\frac{C}{r}\left(-\frac{K}{r^2}\right)}+\underbrace{\frac{u_\theta}{r}\cancel{\frac{\partial u_\theta}{\partial\theta}}}_{\frac{K}{r^2}(0)}+\underbrace{\frac{u_r u_\theta}{r}}_{\frac{CK}{r^3}}+\underbrace{u_z\cancel{\frac{\partial u_\theta}{\partial z}}}_{0\text{ (2-D)}}\right)$$
$$=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\underbrace{\cancel{\rho g_\theta}}_{0}+\mu\left(\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)}_{\frac{K}{r^3}}-\underbrace{\frac{u_\theta}{r^2}}_{\frac{K}{r^3}}+\underbrace{\frac{1}{r^2}\cancel{\frac{\partial^2 u_\theta}{\partial\theta^2}}}_{0}-\underbrace{\frac{2}{r^2}\cancel{\frac{\partial u_r}{\partial\theta}}}_{0}+\underbrace{\cancel{\frac{\partial^2 u_\theta}{\partial z^2}}}_{0\text{ (2-D)}}\right)\qquad(1)$$

The θ momentum equation reduces to

θ momentum:
$$\frac{\partial P}{\partial\theta}=0\qquad(2)$$

The θ momentum equation is satisfied provided we can generate a pressure field that satisfies Eq. 2. As a side note, we might have expected Eq. 2 without even working through the algebra, since in this problem the velocity field is independent of angle θ; we expect that pressure does not depend on θ either. In similar fashion the r momentum equation is

$$\rho\left(\underbrace{\cancel{\frac{\partial u_r}{\partial t}}}_{0\text{ (steady)}}+\underbrace{u_r\frac{\partial u_r}{\partial r}}_{\frac{C}{r}\left(\frac{-C}{r^2}\right)}+\underbrace{\frac{u_\theta}{r}\cancel{\frac{\partial u_r}{\partial\theta}}}_{\frac{K}{r^2}(0)}-\underbrace{\frac{u_\theta^{\,2}}{r}}_{\frac{K^2}{r^3}}+\underbrace{u_z\cancel{\frac{\partial u_r}{\partial z}}}_{0\text{ (2-D)}}\right)$$
$$=-\frac{\partial P}{\partial r}+\underbrace{\cancel{\rho g_r}}_{0}+\mu\left(\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)}_{\frac{C}{r^3}}-\underbrace{\frac{u_r}{r^2}}_{\frac{C}{r^3}}+\underbrace{\frac{1}{r^2}\cancel{\frac{\partial^2 u_r}{\partial\theta^2}}}_{0}-\underbrace{\frac{2}{r^2}\cancel{\frac{\partial u_\theta}{\partial\theta}}}_{0}+\underbrace{\cancel{\frac{\partial^2 u_r}{\partial z^2}}}_{0\text{ (2-D)}}\right)$$

which reduces to

r momentum:
$$\frac{\partial P}{\partial r}=\rho\frac{K^2+C^2}{r^3}\qquad(3)$$

The r momentum equation is satisfied provided we can generate a pressure field that satisfies Eq. 3.

The pressure field $P(r,\theta)$ must be a smooth function of r and θ. Mathematically, this requires that the order of differentiation (r then θ versus θ then r) should not matter. We therefore check whether this is so by differentiating Eqs. 2 and 3 respectively:

Cross-differentiation: $$\frac{\partial^2 P}{\partial r \partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial P}{\partial \theta}\right) = 0 \qquad \frac{\partial^2 P}{\partial \theta \partial r} = \frac{\partial}{\partial \theta}\left(\frac{\partial P}{\partial r}\right) = 0 \qquad (4)$$

Equation 4 shows that indeed, P is a smooth function of r and θ. Thus, *we should be able to calculate the pressure field*.

To calculate $P(r,\theta)$, we start with either Eq. 2 or Eq. 3 and integrate. We pick Eq. 2, which we can partially integrate (with respect to θ) to obtain an expression for $P(r,\theta)$,

Pressure field from θ-momentum: $$P(r,\theta) = 0 + g(r) \qquad (5)$$

Note that we added an arbitrary function of the other variable r rather than a constant of integration since this is a partial integration. We then take the partial derivative of Eq. 5 with respect to r to obtain

$$\frac{\partial P}{\partial r} = g'(r) = \rho \frac{K^2 + C^2}{r^3} \qquad (6)$$

where we have equated our result to Eq. 3 for consistency. We can now integrate Eq. 6 to obtain the function $g(r)$:

$$g(r) = -\frac{1}{2}\rho \frac{K^2 + C^2}{r^2} + C_1 \qquad (7)$$

where C_1 is an arbitrary constant of integration. Finally, we plug Eq. 7 into Eq. 5 to obtain our final expression for $P(x,y)$. The result is

Answer: $$\boxed{P(r,\theta) = -\frac{1}{2}\rho \frac{K^2 + C^2}{r^2} + C_1} \qquad (8)$$

Thus the pressure field for this flow decreases like $1/r^2$ as we approach the origin. (The origin itself is a singularity point.) This flow field is a simplistic model of a tornado or hurricane, and the low pressure at the center is the "eye of the storm". We note that this flow field is irrotational, and thus Bernoulli's equation can be used instead to calculate the pressure. If we call the pressure P_∞ far away from the origin ($r \to \infty$), where the local velocity approaches zero, Bernoulli's equation shows that at any distance r from the origin,

Bernoulli equation: $$P + \frac{1}{2}\rho V^2 = P_\infty \qquad P = P_\infty - \frac{1}{2}\rho \frac{K^2 + C^2}{r^2} \qquad (9)$$

Equation 9 agrees with our solution (Eq. 8) from the full Navier-Stokes equation if we set constant C_1 equal to P_∞. A region of rotational flow near the origin would avoid the singularity there, and would yield a more physically realistic model of a real tornado.

Discussion For practice, try to obtain Eq. 8 by starting with Eq. 3 rather than Eq. 2; you should get the same answer.

9-90

Solution For a given velocity field, we are to calculate the pressure field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** Gravity does not act in either the *x* or the *y* direction.

Analysis The flow field must satisfy the steady, two-dimensional, incompressible continuity *and* momentum equations. We check each equation separately; let's consider continuity first:

Continuity:
$$\underbrace{\frac{\partial u}{\partial x}}_{a}+\underbrace{\frac{\partial v}{\partial y}}_{-a}+\underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0\text{ (2-D)}}=0$$

Continuity is satisfied. Now we look at the *x component* of the Navier-Stokes equation:

x momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0\text{ (steady)}}+\underbrace{u\frac{\partial u}{\partial x}}_{(ax+b)a}+\underbrace{\cancel{v\frac{\partial u}{\partial y}}}_{(-ay+cx^2)0}+\underbrace{\cancel{w\frac{\partial u}{\partial z}}}_{0\text{ (2-D)}}\right)=-\frac{\partial P}{\partial x}+\underbrace{\cancel{\rho g_x}}_{0}+\mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0}+\underbrace{\cancel{\frac{\partial^2 u}{\partial y^2}}}_{0}+\underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{0\text{ (2-D)}}\right) \qquad (1)$$

Equation 1 reduces to

x momentum:
$$\frac{\partial P}{\partial x}=\rho\left(-a^2x-ab\right) \qquad (2)$$

The *x* momentum equation is satisfied provided we can generate a pressure field that satisfies Eq. 2. In similar fashion we examine the *y* momentum equation,

y momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{0\text{ (steady)}}+\underbrace{u\frac{\partial v}{\partial x}}_{(ax+b)2cx}+\underbrace{v\frac{\partial v}{\partial y}}_{(-ay+cx^2)(-a)}+\underbrace{\cancel{w\frac{\partial v}{\partial z}}}_{0\text{ (2-D)}}\right)=-\frac{\partial P}{\partial y}+\underbrace{\cancel{\rho g_y}}_{0}+\mu\left(\underbrace{\frac{\partial^2 v}{\partial x^2}}_{2c}+\underbrace{\cancel{\frac{\partial^2 v}{\partial y^2}}}_{0}+\underbrace{\cancel{\frac{\partial^2 v}{\partial z^2}}}_{0\text{ (2-D)}}\right)$$

The *y* momentum equation reduces to

y momentum:
$$\frac{\partial P}{\partial y}=\rho\left(-acx^2-2bcx-a^2y\right)+2c\mu \qquad (3)$$

The *y* momentum equation is satisfied provided we can generate a pressure field that satisfies Eq. 3.

In the two-dimensional flow under discussion here, the pressure field $P(x,y)$ must be a smooth function of *x* and *y*. Mathematically, this requires that the order of differentiation (*x* then *y* versus *y* then *x*) should not matter. We check whether this is so by differentiating Eqs. 3 and 2 respectively:

Cross-differentiation:
$$\frac{\partial^2 P}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right)=0 \qquad \frac{\partial^2 P}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)=\rho\left(-2acx-2bc\right) \qquad (4)$$

Since the cross-derivative terms in Eq. 4 do not match, *P* is *not* a smooth function of *x* and *y*. Thus, **we are unable to calculate a steady, incompressible, two-dimensional pressure field with the given velocity field**. We cannot proceed any farther – **the pressure cannot be found** with the given velocity field and restrictions.

Discussion This problem shows that if a velocity field satisfies the continuity equation (conservation of mass), this does not necessarily guarantee that the velocity field is physically possible. In the present case, for instance, we are unable to find a pressure field that can satisfy the steady form of the Navier-Stokes equation.

9-91

Solution For a given geometry and set of boundary conditions, we are to calculate the velocity field, and plot the nondimensionalized velocity profile.

Assumptions We number and list the assumptions for clarity:

1 The walls are infinite in the y-z plane (y is into the page).
2 The flow is steady, i.e. time derivatives of any quantity are zero.
3 The flow is parallel (the x component of velocity, u, is zero everywhere).
4 The fluid is incompressible and Newtonian, and the flow is laminar.
5 Pressure P = constant everywhere. In other words, there is no applied pressure gradient pushing the flow; the flow establishes itself due to a balance between gravitational forces and viscous forces.
6 The velocity field is purely two-dimensional, which implies that $v = 0$ and all y derivatives are zero.
7 Gravity acts in the negative z direction. We can express this mathematically as $\vec{g} = -g\vec{k}$, or $g_x = g_y = 0$ and $g_z = -g$.

Analysis We obtain the velocity and pressure fields by following the step-by-step procedure for differential fluid flow solutions.

Step 1 Set up the problem and the geometry. See problem statement.

Step 2 List assumptions and boundary conditions. We have already listed seven assumptions. The boundary conditions come from the no-slip condition at the walls (1) at $x = -h/2$, $u = v = w = 0$. (2) At $x = h/2$, $u = v = w = 0$.

Step 3 Write out and simplify the differential equations. We start with the continuity equation in Cartesian coordinates,

Continuity:
$$\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{\text{Assumption 6}} + \frac{\partial w}{\partial z} = 0 \quad \text{or} \quad \frac{\partial w}{\partial z} = 0 \qquad (1)$$

Equation 1 tells us that w is not a function of z. In other words, it doesn't matter where we place our origin – the flow is the same at any z location. In other words the flow is *fully developed*. Since w is not a function of time (Assumption 2), z (Eq. 1), or y (Assumption 6), we conclude that w is at most a function of x,

Result of continuity:
$$w = w(x) \text{ only} \qquad (2)$$

We now simplify each component of the Navier-Stokes equation as far as possible. Since $u = v = 0$ everywhere and gravity does not act in the x or y directions, the x and y momentum equations are satisfied exactly (in fact all terms are zero in both equations). The z momentum equation reduces to

z momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial w}{\partial t}}}_{\text{Assumption 2}} + \underbrace{u\cancel{\frac{\partial w}{\partial x}}}_{\text{Assumption 3}} + \underbrace{v\cancel{\frac{\partial w}{\partial y}}}_{\text{Assumption 6}} + \underbrace{w\cancel{\frac{\partial w}{\partial z}}}_{\text{Continuity}}\right) = \underbrace{-\cancel{\frac{\partial P}{\partial z}}}_{\text{Assumption 5}} + \rho\underbrace{g_z}_{-\rho g}$$
$$+\mu\left(\frac{\partial^2 w}{\partial x^2} + \underbrace{\cancel{\frac{\partial^2 w}{\partial y^2}}}_{\text{Assumption 6}} + \underbrace{\cancel{\frac{\partial^2 w}{\partial z^2}}}_{\text{continuity}}\right) \quad \text{or} \quad \frac{d^2 w}{dx^2} = \frac{\rho g}{\mu} \qquad (3)$$

We have changed from a partial derivative ($\partial/\partial x$) to a total derivative (d/dx) in Eq. 3 as a direct result of Eq. 2, reducing the PDE to an ODE.

Step 4 Solve the differential equations. Continuity and x and y momentum have already been "solved". Equation 3 (z momentum) is integrated twice to get

Integration of z momentum:
$$w = \frac{\rho g}{2\mu}x^2 + C_1 x + C_2 \qquad (4)$$

Step 5 We apply boundary conditions (1) and (2) from Step 2 above to obtain constants C_1 and C_2, *Boundary condition* (1): $$0 = \frac{\rho g}{8\mu} h^2 - C_1 \frac{h}{2} + C_2$$

and

Boundary condition (2): $$0 = \frac{\rho g}{8\mu} h^2 + C_1 \frac{h}{2} + C_2$$

We solve the above two equations simultaneously to obtain expressions for C_1 and C_2,

Constants of integration: $$C_1 = 0 \qquad C_2 = \frac{-\rho g}{8\mu} h^2$$

Finally, Eq. 4 becomes

Final result for velocity field: $$w = \frac{\rho g}{2\mu}\left(x^2 - \left(\frac{h}{2}\right)^2 \right) \qquad (5)$$

Since $-h/2 < x < h/2$ everywhere, w is negative everywhere as expected (flow is downward).

Step 6 Verify the results. You can plug in the velocity field to verify that all the differential equations and boundary conditions are satisfied.

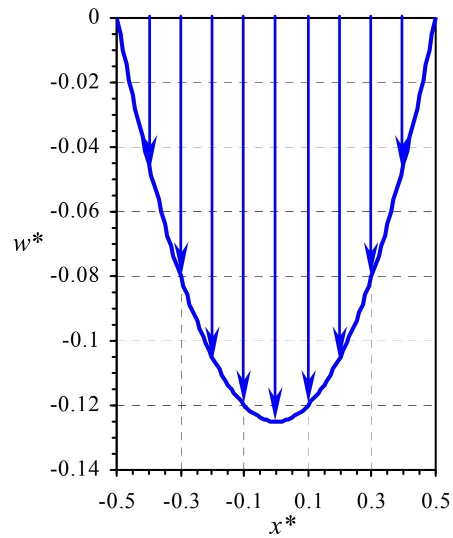


FIGURE 1
The velocity profile for liquid falling between two vertical walls.

We nondimensionalize Eq. 5 by inspection: we let $x^* = x/h$ and $w^* = w\mu/(\rho g h^2)$. Eq. 5 becomes

Nondimensionalized velocity profile: $$\boxed{w^* = \frac{1}{2}\left((x^*)^2 - \frac{1}{4} \right)} \qquad (6)$$

We plot the nondimensional velocity field in Fig. 1. The velocity profile is parabolic.

Discussion Equation 4 for the z component of velocity is identical to that of Example 9-17. In fact, the present problem is identical to Example 9-17 except for the boundary conditions and the location of the origin. Comparing the two results, we see that the maximum nondimensional velocity for the case with two walls is one-fourth than that for the case with only one wall. This is not unexpected – the additional wall leads to more viscous forces that retard the flow.

9-92

Solution We are to calculate and compare the volume flow rate per unit width of fluid falling between two vertical walls and fluid falling along one vertical wall.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The walls are infinitely wide and very long so that all of the parallel flow, fully developed approximations of the previous problem hold.

Analysis We calculate the volume flow rate per unit width by integration of the velocity:

Volume flow rate per unit depth, two vertical walls:

$$\boxed{\frac{\dot{V}}{L}=\int_{-h/2}^{h/2} w dx=\int_{-h/2}^{h/2}\left[\frac{\rho g}{2\mu}\left(x^2-\left(\frac{h}{2}\right)^2\right)\right]dx=\frac{\rho g}{2\mu}\left[\frac{x^3}{3}-\frac{h^2}{4}x\right]_{x=-h/2}^{x=h/2}=\frac{\rho g}{2\mu}\left[\frac{h^3}{24}-\frac{h^3}{8}+\frac{h^3}{24}-\frac{h^3}{8}\right]=\frac{-\rho g h^3}{12\mu}} \quad (1)$$

The result is negative since we have defined positive volume flow rate upward, since z is upward, but the flow is downward. For the case with only one vertical wall and a free surface, we calculate the vertical component of velocity to be $w=\frac{\rho g x}{2\mu}(x-2h)$ (see Example 9-17). Thus, we calculate $\dot{V}/L$ for the case of one vertical wall to be

Volume flow rate per unit depth, one vertical wall with a free surface:

$$\boxed{\frac{\dot{V}}{L}=\int_0^h w dx=\int_0^h\left[\frac{\rho g x}{2\mu}(x-2h)\right]dx=\frac{\rho g}{2\mu}\left[\frac{x^3}{3}-x^2 h\right]_{x=0}^{x=h}=\frac{\rho g}{2\mu}\left[\frac{h^3}{3}-h^3-0+0\right]=\frac{-\rho g h^3}{3\mu}} \quad (2)$$

Comparing the two cases we see that $\dot{V}/L$ ***for the case of one vertical wall and a free surface is four times greater than the case of two vertical walls with no free surface***. The physical explanation is that with two walls, the fluid is held back by more viscous stresses, leading to a parabolic velocity profile. For the single-wall case the free surface has no shear stress and thus the fluid flows more freely.

Discussion The two flows being compared here are identical except for the boundary conditions. This illustrates the importance of setting proper boundary conditions.

9-93

Solution For a given geometry and set of boundary conditions, we are to calculate the velocity and pressure fields, and plot the nondimensional velocity profile.

Assumptions We number and list the assumptions for clarity:

1 The wall is infinite in the *s*-*y* plane (*y* is out of the page for a right-handed coordinate system).

2 The flow is steady, i.e. $\frac{\partial}{\partial t}(\text{anything}) = 0$.

3 The flow is parallel and fully developed (we assume the normal component of velocity, u_n, is zero, and we assume that the streamwise component of velocity u_s is independent of streamwise coordinate *s*).

4 The fluid is incompressible and Newtonian, and the flow is laminar.

5 Pressure P = constant = P_{atm} at the free surface. In other words, there is no applied pressure gradient pushing the flow; the flow establishes itself due to a balance between gravitational forces and viscous forces along the wall. Atmospheric pressure is constant everywhere since we are neglecting the change of air pressure with elevation.

6 The velocity field is purely two-dimensional, which implies that $v = 0$ and $\frac{\partial}{\partial y}$ (any velocity component) = 0.

7 Gravity acts in the negative *z* direction. We can express this mathematically as $\vec{g} = -g\vec{k}$. In the *s*-*n* plane, $g_s = g\sin\alpha$ and $g_n = -g\cos\alpha$.

Analysis We obtain the velocity and pressure fields by following the step-by-step procedure for differential fluid flow solutions.

Step 1 Set up the problem and the geometry. See Problem statement.

Step 2 List assumptions and boundary conditions. We have already listed seven assumptions. The boundary conditions are (1) No slip at the wall: at $n = 0$, $u_s = v = u_n = 0$. (2) At the free surface ($n = h$) there is no shear, which in this coordinate system at the vertical free surface means $\partial u_s/\partial n = 0$. (3) $P = P_{\text{atm}}$ at $n = h$.

Step 3 Write out and simplify the differential equations. We start with the continuity equation in modified Cartesian coordinates, (s,y,n) and (u_s,v,u_n),

Continuity:
$$\frac{\partial u_s}{\partial s} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{\text{Assumption 6}} + \underbrace{\cancel{\frac{\partial u_n}{\partial n}}}_{\text{Assumption 3}} = 0 \qquad \text{or} \qquad \frac{\partial u_s}{\partial s} = 0 \qquad (1)$$

Equation 1 tells us that u_s is not a function of *s*. In other words, it doesn't matter where we place our origin – the flow is the same at any *s* location. This does not tell us anything new; we have already assumed that the flow is fully developed (Assumption 3). Furthermore, since u_s is not a function of time (Assumption 2) or *y* (Assumption 6), we conclude that u_s is at most a function of *n*,

Result of continuity:
$$u_s = u_s(n) \text{ only} \qquad (2)$$

We now simplify each component of the Navier-Stokes equation as far as possible. Since $v = 0$ everywhere and gravity does not act in the *y* direction, the *y* momentum equation is satisfied exactly (in fact all terms are zero). Since $u_n = 0$ everywhere, the only non-zero terms in the *n* momentum equation are the pressure term and the gravity term. The *n* momentum equation reduces to

n momentum:
$$\rho \underbrace{\cancel{\frac{Du_n}{Dt}}}_{\text{Assumption 3}} = -\frac{\partial P}{\partial n} + \underbrace{\rho g_n}_{-\rho g\cos\alpha} + \mu \underbrace{\cancel{\nabla^2 u_n}}_{\text{Assumption 3}} \qquad \text{or} \qquad \frac{\partial P}{\partial n} = -\rho g\cos\alpha \qquad (3)$$

We integrate Eq. 3 to solve for the pressure,

Pressure:
$$P = -\rho g n\cos\alpha + f(s) \qquad (4)$$

where we have added a function of *s* rather than a simple constant of integration. But from boundary condition (3), at $n = h$, $P = P_{\text{atm}}$. Thus Eq. 4 yields $f(s) = P_{\text{atm}} + \rho g h\cos\alpha$. In other words, $f(s)$ is really not a function of *s* at all. Equation 4 then becomes

Final expression for pressure: $\boxed{P = P_{\text{atm}} + \rho g (h-n)\cos\alpha}$ (5)

The s momentum equation reduces to

$$\rho\left(\underbrace{\cancel{\frac{\partial u_s}{\partial t}}}_{\text{Assumption 2}} + \underbrace{u_s\cancel{\frac{\partial u_s}{\partial s}}}_{\text{Continuity}} + \underbrace{v\cancel{\frac{\partial u_s}{\partial y}}}_{\text{Assumption 6}} + \underbrace{u_n\cancel{\frac{\partial u_s}{\partial n}}}_{\text{Assumption 3}}\right) = \underbrace{-\cancel{\frac{\partial P}{\partial s}}}_{\text{Eq. 5}} + \underbrace{\rho g_s}_{\rho g\sin\alpha}$$
$$+\mu\left(\underbrace{\cancel{\frac{\partial^2 u_s}{\partial s^2}}}_{\text{continuity}} + \underbrace{\cancel{\frac{\partial^2 u_s}{\partial y^2}}}_{\text{Assumption 6}} + \frac{\partial^2 u_s}{\partial n^2}\right) \quad \text{or} \quad \frac{d^2 u_s}{dn^2} = -\frac{\rho g\sin\alpha}{\mu} \tag{6}$$

We have changed from a partial derivative ($\partial/\partial n$) to a total derivative (d/dn) in Eq. 6 as a direct result of Eq. 2, reducing the PDE to an ODE.

Step 4 Solve the differential equations. Continuity and n and y momentum have already been "solved". Equation 6 (s momentum) is integrated twice to get

$$u_s = -\frac{\rho g\sin\alpha}{2\mu}n^2 + C_1 n + C_2 \tag{7}$$

Step 5 We apply boundary conditions (1) and (2) from Step 2 above to obtain constants C_1 and C_2,

Boundary condition (1): $u_s = 0+0+C_2$ at $n=0$ $\quad C_2 = 0$

and

Boundary condition (2):

$$\left.\frac{du_s}{dn}\right)_{n=h} = -\frac{\rho g\sin\alpha}{\mu}h + C_1 = 0 \qquad C_1 = \frac{\rho g h\sin\alpha}{\mu}$$

Finally, Eq. 4 becomes

Final result for velocity field: $\boxed{u_s = \frac{\rho g\sin\alpha}{2\mu}n(2h-n)}$ (8)

Since $n < h$ in the film, u_s is positive everywhere as expected (flow is downward).

Step 6 Verify the results. You can plug in the velocity field to verify that all the differential equations and boundary conditions are satisfied.

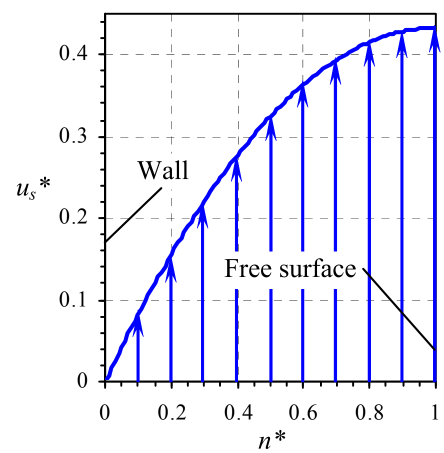


FIGURE 1
The velocity profile for an oil film falling down an inclined wall, $\alpha = 60^{\circ}$.

When $\alpha = 90^{\circ}$ $\sin\alpha = 1$ and Eq. 8 is equivalent to Eq. 5 of Example 9-17. (The signs are opposite since s is down while z is up.) Also, Eq. 5 above reduces to $P = P_{\text{atm}}$ everywhere when $\alpha = 90^{\circ}$ since $\cos\alpha = 0$; this also agrees with the results of Example 9-17. We nondimensionalize Eq. 8 by inspection: we let $n^* = n/h$ and $u_s^* = u_s\mu/(\rho g h^2)$. Eq. 8 becomes

Nondimensional velocity profile: $\boxed{u_s^* = \frac{n^*}{2}(2-n^*)\sin\alpha}$ (9)

We plot the nondimensional velocity field in Fig. 1 for the case in which $\alpha = 60^{\circ}$.

Discussion The profile shape is identical to that of Example 9-17, but is scaled by the factor $\sin\alpha$. This problem could also have been solved in standard Cartesian coordinates (x,y,z), but the algebra would be more involved.

9-94

Solution We are to calculate the volume flow rate per unit width of oil falling down a vertical wall.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The wall is infinitely wide and very long so that all of the parallel flow, fully developed approximations of Problem 9-89 hold.

Analysis We calculate the volume flow rate per unit width by integration of the velocity:

Volume flow rate per unit depth:
$$\boxed{\frac{\dot{V}}{L} = \int_0^h u_s dn = \int_0^h \left[\frac{\rho g \sin\alpha}{2\mu} n(2h-n)\right] dn = \frac{\rho g \sin\alpha}{3\mu} h^3} \qquad (1)$$

For an oil film of thickness 5.0 mm with ρ = 888 kg/m^3 and μ = 0.80 kg/(m·s), we calculate $\dot{V}/L$ using Eq. 1,

Result:
$$\frac{\dot{V}}{L} = \frac{\rho g \sin\alpha}{3\mu} h^3 = \frac{\left(888 \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)\sin\left(60^\circ\right)\left(0.005 \text{ m}\right)^3}{3\left(0.80 \text{ kg/m}\cdot\text{s}\right)} = \mathbf{3.93\times10^{-4}\ m^2/s}$$

Discussion Since viscosity is in the denominator of Eq. 1, a low viscosity liquid (like water) would yield a larger volume flow rate; this agrees with our intuition. Likewise, a larger density liquid and/or a thicker film would yield a larger volume flow rate, again agreeing with our intuition. Finally, if $\alpha = 0^\circ$ there is no flow.

9-95

Solution We are to expand two terms into three terms, and then compress the three terms into one term.

Analysis We use the product rule to differentiate the expression,

$$\boxed{\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2}\right) = \mu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right)}$$

The second part of this question involves some trial and error, using the product rule in reverse. After some effort we get

$$\boxed{\mu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right) = \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r u_\theta\right)\right)\right)} \qquad (1)$$

You can apply the product rule to verify Eq. 1.

Discussion The grouping of these terms into one term as in Eq. 1 turns out to be useful for some analytical solutions of the Navier-Stokes equation.

9-96

Solution For a given geometry and set of boundary conditions, we are to calculate the velocity field.

Assumptions We number and list the assumptions for clarity:

1 The cylinders are infinite in the z direction (z is out of the page in the figure of the problem statement for a right-handed coordinate system). The velocity field is purely two-dimensional, which implies that $w = 0$ and derivatives of any velocity component with respect to z are zero.
2 The flow is steady, meaning that all time derivatives are zero.
3 The flow is circular, meaning that the radial velocity component u_r is zero.
4 The flow is rotationally symmetric, meaning that nothing is a function of θ.
5 The fluid is incompressible and Newtonian, and the flow is laminar.
6 Gravitational effects are ignored. (Note that gravity may act in the z direction, leading to an additional hydrostatic pressure distribution in the z direction. This would not affect the present analysis.)

Analysis We obtain the velocity and pressure fields by following the step-by-step procedure for differential fluid flow solutions.

Step 1 Set up the problem and the geometry. See the problem statement.
Step 2 List assumptions and boundary conditions. We have already listed five assumptions. The boundary conditions are (1) No slip at the inner wall: at $r = R_i$, $u_\theta = \omega_i R_i$. (2) No slip at the outer wall: at $r = R_o$, $u_\theta = 0$.
Step 3 Write out and simplify the differential equations. We start with the continuity equation in cylindrical coordinates, (r,θ,z) and (u_r,u_θ,u_z),

Continuity:
$$\underbrace{\cancel{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}}_{\text{Assumption 4}} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{\text{Assumption 1}} = 0 \qquad \text{or} \qquad 0 = 0 \qquad (1)$$

Thus continuity is satisfied exactly by our assumptions.

We now simplify each component of the Navier-Stokes equation as far as possible. Since $w = 0$ everywhere and gravity is ignored, the z momentum equation is satisfied exactly (in fact all terms are zero). Since $u_r = 0$ everywhere, the only non-zero terms in the r momentum equation are the pressure term and the "extra" term that involves u_θ. The r momentum equation reduces to

r momentum:
$$\frac{\partial P}{\partial r} = \rho\frac{u_\theta^{\,2}}{r} \qquad \text{or} \qquad \frac{dP}{dr} = \rho\frac{u_\theta^{\,2}}{r} \qquad (2)$$

We have changed the partial derivatives to total derivatives since P is a function only of r. Equation 2 could be used to solve for $P(r)$ once we find u_θ.

The θ momentum equation is written out, using the result of the previous problem,

θ momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial u_\theta}{\partial t}}}_{\text{Assumption 2}} + \underbrace{\cancel{u_r\frac{\partial u_\theta}{\partial r}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}}}_{\text{Assumption 4}} + \underbrace{\cancel{\frac{u_r u_\theta}{r}}}_{\text{Assumption 3}} + \underbrace{\cancel{u_z\frac{\partial u_\theta}{\partial z}}}_{\text{Assumption 1}}\right)$$
$$= -\underbrace{\cancel{\frac{1}{r}\frac{\partial P}{\partial \theta}}}_{\text{Assumption 4}} + \underbrace{\cancel{\rho g_\theta}}_{\text{Assumption 6}} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}}}_{\text{Assumption 4}} - \underbrace{\cancel{\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{\partial^2 u_\theta}{\partial z^2}}}_{\text{Assumption 6}}\right)$$

Again we change from partial derivatives ($\partial/\partial r$) to a total derivatives (d/dr), reducing the PDE to an ODE. The θ momentum equation reduces to

Reduced θ momentum:
$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru_\theta)\right) = 0 \qquad (3)$$

Step 4 Solve the differential equations. Continuity and z momentum have already been "solved". Equation 3 (θ momentum) is integrated once, *Integration of θ momentum*: $$\frac{1}{r}\frac{d}{dr}\left(ru_\theta\right) = C_1$$

After multiplying by *r* we integrate again. After division by *r* we get

u_θ: $$u_\theta = C_1\frac{r}{2} + \frac{C_2}{r} \qquad (4)$$

Step 5 We apply boundary conditions (1) and (2) from Step 2 above to obtain constants C_1 and C_2,

Boundary condition (2): $$0 = C_1\frac{R_o}{2} + \frac{C_2}{R_o} \quad \text{or} \quad C_2 = -C_1\frac{R_o^{\,2}}{2}$$

and

Boundary condition (1): $$R_i\omega_i = C_1\frac{R_i}{2} + \frac{C_2}{R_i} = C_1\frac{R_i}{2} - C_1\frac{R_o^{\,2}}{2R_i}$$

Which can be solved for C_1. The two constants of integration are thus

Constants of integration: $$C_1 = \frac{-2R_i^{\,2}\omega_i}{R_o^{\,2} - R_i^{\,2}} \qquad C_2 = \frac{R_o^{\,2}R_i^{\,2}\omega_i}{R_o^{\,2} - R_i^{\,2}}$$

Finally, Eq. 4 becomes (after a bit of algebra)

Final result for velocity field: $$\boxed{u_\theta = \frac{R_i^{\,2}\omega_i}{R_o^{\,2} - R_i^{\,2}}\left(\frac{R_o^{\,2}}{r} - r\right)} \qquad (5)$$

Step 6 Verify the results. You can plug in the velocity field to verify that all the differential equations and boundary conditions are satisfied.

Discussion There are valid alternative forms of Eq. 5. We could integrate Eq. 2 to solve for the pressure since we now know u_θ from Eq. 5. The algebra is laborious, but not difficult.

9-97

Solution For a given geometry and set of boundary conditions, we are to calculate the velocity field.

Assumptions We number and list the assumptions for clarity:

1 The cylinders are infinite in the z direction (z is out of the page in the figure of the problem statement for a right-handed coordinate system). The velocity field is purely two-dimensional, which implies that $w = 0$ and derivatives of any velocity component with respect to z are zero.
2 The flow is steady, meaning that all time derivatives are zero.
3 The flow is circular, meaning that the radial velocity component u_r is zero.
4 The flow is rotationally symmetric, meaning that nothing is a function of θ.
5 The fluid is incompressible and Newtonian, and the flow is laminar.
6 Gravitational effects are ignored. (Note that gravity may act in the z direction, leading to an additional hydrostatic pressure distribution in the z direction. This would not affect the present analysis.)

Analysis We obtain the velocity and pressure fields by following the step-by-step procedure for differential fluid flow solutions.

Step 1 Set up the problem and the geometry. See the problem statement.

Step 2 List assumptions and boundary conditions. We have already listed five assumptions. The boundary conditions are (1) No slip at the inner wall: at $r = R_i$, $u_\theta = 0$. (2) No slip at the outer wall: at $r = R_o$, $u_\theta = \omega_o R_o$.

Step 3 Write out and simplify the differential equations. We start with the continuity equation in cylindrical coordinates, (r,θ,z) and (u_r,u_θ,u_z),

Continuity:
$$\underbrace{\cancel{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}}_{\text{Assumption 4}} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{\text{Assumption 1}} = 0 \qquad \text{or} \qquad 0 = 0 \qquad (1)$$

Thus continuity is satisfied exactly by our assumptions.

We now simplify each component of the Navier-Stokes equation as far as possible. Since $w = 0$ everywhere and gravity is ignored, the z momentum equation is satisfied exactly (in fact all terms are zero). Since $u_r = 0$ everywhere, the only non-zero terms in the r momentum equation are the pressure term and the "extra" term that involves u_θ. The r momentum equation reduces to

r momentum:
$$\frac{\partial P}{\partial r} = \rho\frac{u_\theta^{\;2}}{r} \qquad \text{or} \qquad \frac{dP}{dr} = \rho\frac{u_\theta^{\;2}}{r} \qquad (2)$$

We have changed the partial derivatives to total derivatives since P is a function only of r. Equation 2 could be used to solve for $P(r)$ once we find u_θ.

The θ momentum equation is written out, using the result of the previous problem,

θ momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial u_\theta}{\partial t}}}_{\text{Assumption 2}} + \underbrace{\cancel{u_r\frac{\partial u_\theta}{\partial r}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}}}_{\text{Assumption 4}} + \underbrace{\cancel{\frac{u_r u_\theta}{r}}}_{\text{Assumption 3}} + \underbrace{\cancel{u_z\frac{\partial u_\theta}{\partial z}}}_{\text{Assumption 1}}\right)$$
$$= -\underbrace{\cancel{\frac{1}{r}\frac{\partial P}{\partial \theta}}}_{\text{Assumption 4}} + \underbrace{\cancel{\rho g_\theta}}_{\text{Assumption 6}} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}}}_{\text{Assumption 4}} - \underbrace{\cancel{\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{\partial^2 u_\theta}{\partial z^2}}}_{\text{Assumption 6}}\right)$$

Again we change from partial derivatives ($\partial/\partial r$) to a total derivatives (d/dr), reducing the PDE to an ODE. The θ momentum equation reduces to

Reduced θ momentum:
$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru_\theta)\right) = 0 \qquad (3)$$

Step 4 Solve the differential equations. Continuity and z momentum have already been "solved". Equation 3 (θ momentum) is integrated once, *Integration of θ momentum*: $$\frac{1}{r}\frac{d}{dr}\left(ru_\theta\right) = C_1$$

After multiplying by r we integrate again. After division by r we get

u_θ: $$u_\theta = C_1\frac{r}{2} + \frac{C_2}{r} \qquad (4)$$

Step 5 We apply boundary conditions (1) and (2) from Step 2 above to obtain constants C_1 and C_2,

Boundary condition (1): $$0 = C_1\frac{R_i}{2} + \frac{C_2}{R_i} \quad \text{or} \quad C_2 = -C_1\frac{R_i^2}{2}$$

and

Boundary condition (2): $$R_o\omega_o = C_1\frac{R_o}{2} + \frac{C_2}{R_o} = C_1\frac{R_o}{2} - C_1\frac{R_i^2}{2R_o}$$

Which can be solved for C_1. The two constants of integration are thus

Constants of integration: $$C_1 = \frac{2R_o^{\,2}\omega_o}{R_o^{\,2} - R_i^{\,2}} \qquad C_2 = \frac{-R_o^{\,2}R_i^{\,2}\omega_o}{R_o^{\,2} - R_i^{\,2}}$$

Finally, Eq. 4 becomes (after a bit of algebra)

Final result for velocity field: $$\boxed{u_\theta = \frac{R_o^{\,2}\omega_o}{R_o^{\,2} - R_i^{\,2}}\left(r - \frac{R_i^{\,2}}{r}\right)} \qquad (5)$$

Step 6 Verify the results. You can plug in the velocity field to verify that all the differential equations and boundary conditions are satisfied.

Discussion There are valid alternative forms of Eq. 5. We could integrate Eq. 2 to solve for the pressure since we now know u_θ from Eq. 5. The algebra is laborious, but not difficult.

9-98

Solution We are to simplify the velocity field for two limiting cases of Problem 9-96 and discuss.

Assumptions The same assumptions of Problem 9-96 apply here.

Analysis (*a*) First we re-write the velocity profile from Problem 9-92,

Exact velocity profile:

$$u_\theta = \frac{R_i^2\omega_i}{R_o^2 - R_i^2}\left(\frac{R_o^2}{r} - r\right) = \frac{R_i^2\omega_i}{(R_o - R_i)(R_o + R_i)}\left(\frac{(R_o - r)(R_o + r)}{r}\right) \qquad (1)$$

Note that Eq. 1 is still exact. When the gap is very small, $(R_o - R_i) << R_o$, and $R_o \approx R_i$. Thus we replace $R_o + R_i$ in the denominator of Eq. 1 by $2R_i$. Similarly, $r \approx R_i$ and we replace $R_o + r$ in the numerator of Eq. 1 by $2R_i$. Likewise we replace r in the denominator of Eq. 1 by R_i. As suggested we define $y = R_o - r$, h = gap thickness = $R_o - R_i$, and V = speed of the "upper plate" = $R_i\omega_i$ (Fig. 1). Plugging all of these approximations and definitions into Eq. 1 we get

Approximate velocity for small gap:

$$\boxed{u_\theta \approx \frac{R_i^2\omega_i}{h \cdot 2R_i}\left(\frac{y \cdot 2R_i}{R_i}\right) = \frac{y\omega_i R_i}{h} = V\frac{y}{h}} \qquad (2)$$

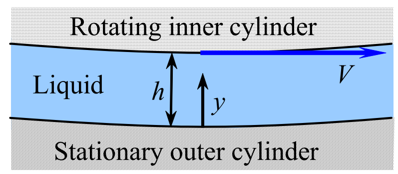


FIGURE 1
A magnified view near the bottom for the case in which the gap between the two cylinders is very small.

We verify that Eq. 2 is linear in the small gap and is the same velocity profile as we generated for 2-D Couette flow between two infinite flat plates.

(*b*) As the outer cylinder radius approaches infinity, $R_i << R_o$, and R_i can be ignored when added to or subtracted from R_o. Similarly, $r << R_o$, and r can be ignored when added to or subtracted from R_o. Equation 1 becomes

Approximate velocity for infinite R_o:

$$\boxed{u_\theta \approx \frac{R_i^2\omega_i}{(R_o)(R_o)}\left(\frac{(R_o)(R_o)}{r}\right) = \frac{R_i^2\omega_i}{r}} \qquad (3)$$

We recognize Eq. 3 as of the form u_θ = constant/r which is the velocity field for a **line vortex**.

Discussion Imagine a long, thin cylinder spinning in a vat of liquid. After a long time, the flow field given by Eq. 3 would emerge – basically a line vortex for all radii greater than R_i.

9-99

Solution For a given geometry and set of boundary conditions, we are to calculate the velocity field.

Assumptions The assumptions are identical to those of Problem 9-96.

Analysis We obtain the velocity and pressure fields by following the step-by-step procedure for differential fluid flow solutions. Everything is identical to Problem 9-96 except for the boundary condition at the outer cylinder wall. We re-write boundary condition (2): at $r = R_o$, $u_\theta = \omega_o R_o$. We will not repeat all the algebra associated with the equations of motion. The tangential velocity component is still

u_θ:
$$u_\theta = C_1 \frac{r}{2} + \frac{C_2}{r} \qquad (1)$$

Now we apply boundary conditions (1) and (2) to obtain constants C_1 and C_2,

Boundary condition (1):
$$\frac{R_i}{2} C_1 + \frac{1}{R_i} C_2 = R_i \omega_i \qquad (2)$$

Boundary condition (2):
$$\frac{R_o}{2} C_1 + \frac{1}{R_o} C_2 = R_o \omega_o \qquad (3)$$

We solve Eqs. 2 and 3 simultaneously for C_1 and C_2. The result is

Constants of integration:
$$C_1 = \frac{2\left(R_o^2 \omega_o - R_i^2 \omega_i\right)}{R_o^2 - R_i^2} \qquad C_2 = \frac{R_o^2 R_i^2 \left(\omega_i - \omega_o\right)}{R_o^2 - R_i^2} \qquad (4)$$

Finally, Eq. 4 becomes (after a bit of algebra)

Final result for velocity field:
$$\boxed{u_\theta = \frac{1}{R_o^2 - R_i^2}\left[\left(R_o^2 \omega_o - R_i^2 \omega_i\right) r + \frac{R_o^2 R_i^2 \left(\omega_i - \omega_o\right)}{r}\right]} \qquad (5)$$

We set $\omega_o = 0$ in Eq. 5 to verify that it simplifies to the result of Problem 9-92,

Simplified velocity field:
$$\boxed{u_\theta = \frac{R_i^2 \omega_i}{R_o^2 - R_i^2}\left(\frac{R_o^2}{r} - r\right)} \qquad (6)$$

Discussion There are valid alternative forms of Eq. 5.

9-100

Solution We are to discuss a simplified version of the velocity field of the previous problem.

Assumptions The assumptions are identical to those of the previous problem.

Analysis We set $R_i = \omega_i = 0$ in Eq. 5 of the previous problem. The tangential velocity component simplifies to

Simplified u_θ:

$$u_\theta = \frac{1}{R_o^{\,2}}\left[R_o^{\,2}\omega_o r\right] = \omega_o r \qquad (1)$$

We recognize Eq. 1 as the velocity field for *solid body rotation*. To set up this velocity field in a physical experiment, we would place a cylindrical container of liquid on a rotating table. After a long time, the entire tank, including the liquid, would be in solid body rotation.

Discussion If you imagine flow between the inner and outer cylinders, and then imagine that the inner cylinder stops spinning and shrinks to infinitesimal radius, you can convince yourself that solid body rotation would result.

9-101

Solution For flow in a pipe annulus we are to calculate the velocity field.

Assumptions We number and list the assumptions for clarity:

1. The pipe is infinitely long in the x direction.
2. The flow is steady, i.e. $\frac{\partial}{\partial t}(\text{anything}) = 0$.
3. This is a parallel flow (the r component of velocity, u_r, is zero).
4. The fluid is incompressible and Newtonian, and the flow is laminar.
5. A constant pressure gradient is applied in the x direction such that pressure changes linearly with respect to x according to the given expression.
6. The velocity field is axisymmetric with no swirl, implying that $u_\theta = 0$ and $\frac{\partial}{\partial \theta}(\text{anything}) = 0$.
7. We ignore the effects of gravity.

Analysis We obtain the velocity field by following the step-by-step procedure for differential fluid flow solutions.

Step 1 Lay out the problem and the geometry. See the problem statement.

Step 2 List assumptions and boundary conditions. We have already listed seven assumptions. The boundary conditions come from imposing the no slip condition at both pipe walls: (1) at $r = R_i$, $\vec{V} = 0$. (2) at $r = R_o$, $\vec{V} = 0$.

Step 3 Write out and simplify the differential equations. We start with the continuity equation in cylindrical coordinates,

Continuity:
$$\underbrace{\cancel{\frac{1}{r}\frac{\partial (r u_r)}{\partial r}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{1}{r}\frac{\partial (u_\theta)}{\partial \theta}}}_{\text{Assumption 6}} + \frac{\partial u}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial x} = 0 \tag{1}$$

Equation 1 tells us that u is not a function of x. In other words, it doesn't matter where we place our origin – the flow is the same at any x location. This can also be inferred directly from Assumption 1, which tells us that there is nothing special about any x location since the pipe is infinite in length – the flow is fully developed. Furthermore, since u is not a function of time (Assumption 2) or θ (Assumption 6), we conclude that u is at most a function of r,

Result of continuity:
$$u = u(r) \text{ only} \tag{2}$$

Next we simplify the x momentum equation as far as possible:

x momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{Assumption 2}} + \underbrace{\cancel{u_r\frac{\partial u}{\partial r}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u}{\partial \theta}}}_{\text{Assumption 6}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\text{Continuity}}\right) = -\frac{\partial P}{\partial x} + \underbrace{\cancel{\rho g_x}}_{\text{Assumption 7}} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}}_{\text{Assumption 6}} + \underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{Continuity}}\right) \tag{3}$$

or

Result of x momentum:
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x} \tag{4}$$

Note that we have replaced the partial derivative operators for the u derivatives with total derivative operators because of Eq. 2. Every term in the r momentum equation is zero except the pressure gradient term, forcing that lone term to also be zero,

r momentum:
$$\frac{\partial P}{\partial r} = 0 \tag{5}$$

In other words, P is not a function of r. Since P is also not a function of time (Assumption 2) or θ (Assumption 6), P can be at most a function of x,

Result of r momentum: $$P = P(x) \text{ only} \qquad (6)$$

Therefore we can replace the partial derivative operator for the pressure gradient in Eq. 4 by the total derivative operator since P varies only with x. Finally, all terms of the θ component of the Navier-Stokes equation go to zero.

Step 4 Solve the differential equations. Continuity and r momentum have already been "solved", resulting in Eqs. 2 and 6 respectively. The θ momentum equation has vanished, and thus we are left with Eq. 4 (x momentum). After multiplying both sides by r, we integrate once to obtain

Integration of x momentum: $$r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1 \qquad (7)$$

where C_1 is a constant of integration. Note that the pressure gradient dP/dx is a constant here. After dividing both sides of Eq. 7 by r, we can integrate a second time to get

Second integration of x momentum: $$u = \frac{r^2}{4\mu}\frac{dP}{dx} + C_1 \ln r + C_2 \qquad (8)$$

where C_2 is a second constant of integration.

Step 5 Apply boundary conditions from Step 2 above to obtain constants C_1 and C_2:

Boundary condition (1): $$0 = \frac{R_i^2}{4\mu}\frac{dP}{dx} + C_1 \ln R_i + C_2$$

Boundary condition (2): $$0 = \frac{R_o^2}{4\mu}\frac{dP}{dx} + C_1 \ln R_o + C_2$$

We solve the above two equations simultaneously to find C_1 and C_2,

Constants: $$C_1 = -\frac{\left(R_o^2 - R_i^2\right)}{4\mu \ln\frac{R_o}{R_i}}\frac{dP}{dx} \qquad C_2 = \frac{\left(R_o^2 \ln R_i - R_i^2 \ln R_o\right)}{4\mu \ln\frac{R_o}{R_i}}\frac{dP}{dx}$$

After some algebra and rearrangement, Eq. 7 becomes

Final result for axial velocity: $$\boxed{u = \frac{1}{4\mu}\frac{dP}{dx}\left(r^2 + \frac{R_i^2 \ln\frac{r}{R_o} - R_o^2 \ln\frac{r}{R_i}}{\ln\frac{R_o}{R_i}}\right)} \qquad (9)$$

Step 6 Verify the results. You can plug in the velocity field to verify that all the differential equations and boundary conditions are satisfied.

Discussion There are other valid forms of Eq. 9. For example, after some rearrangement, Eq. 9 can be written as

Alternative form: $$\boxed{u = \frac{1}{4\mu}\frac{dP}{dx}\left(r^2 - R_o^2 - \frac{R_o^2 - R_i^2}{\ln\frac{R_o}{R_i}}\ln\frac{r}{R_o}\right)} \qquad (10)$$

9-102

Solution We are to generate the velocity field for a given flow setup.

Assumptions All assumptions are the same as those of the previous problem except for the fifth one, which we modify here: **5** Pressure P is constant everywhere.

Analysis Most of the algebra is identical to that of the previous problem except that the pressure gradient is zero, making this problem easier. Also, the first boundary condition changes: at $r = R_i$, $u = V$. The x momentum equation reduces to

Result of x momentum:
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0 \qquad (1)$$

After integration, division by r, and a second integration, Eq. 1 yields

x component of velocity:
$$u = C_1 \ln r + C_2 \qquad (2)$$

We apply boundary conditions:

Boundary condition (1):
$$V = C_1 \ln R_i + C_2$$

and

Boundary condition (2):
$$0 = C_1 \ln R_o + C_2$$

We solve the above two equations simultaneously to yield the constants,

Constants of integration:
$$C_1 = \frac{-V}{\ln\dfrac{R_o}{R_i}} \qquad C_2 = \frac{V \ln R_o}{\ln\dfrac{R_o}{R_i}} \qquad (3)$$

and thus Eq. 2 becomes

Result for u:
$$\boxed{u = \frac{V}{\ln\dfrac{R_o}{R_i}}\left(\ln R_o - \ln r\right) = \frac{V \ln\dfrac{R_o}{r}}{\ln\dfrac{R_o}{R_i}}} \qquad (4)$$

Discussion In this and other parallel flow problems, the nonlinear terms in the Navier-Stokes equation drop out, simplifying the problem and enabling an exact analytical solution to be found.

9-103

Solution We are to generate the velocity field for a given flow setup.

Assumptions All assumptions are the same as those of the previous problem.

Analysis The algebra is identical to that of the previous problem except that the boundary conditions are swapped: at $r = R_i$, $u = 0$ and at $r = R_o$, $u = V$. The x momentum equation reduces to

Result of x momentum: $$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0 \qquad (1)$$

After integration, division by r, and a second integration, Eq. 1 yields

x component of velocity: $$u = C_1 \ln r + C_2 \qquad (2)$$

We apply boundary conditions:

Boundary condition (1): $$V = C_1 \ln R_o + C_2$$

and

Boundary condition (2): $$0 = C_1 \ln R_i + C_2$$

We solve the above two equations simultaneously to yield the constants,

Constants of integration: $$C_1 = \frac{-V}{\ln \frac{R_i}{R_o}} \qquad C_2 = \frac{V \ln R_i}{\ln \frac{R_i}{R_o}} \qquad (3)$$

and thus Eq. 2 becomes

Result for u: $$\boxed{u = \frac{V}{\ln \frac{R_i}{R_o}}\left(\ln R_i - \ln r\right) = \frac{V \ln \frac{R_i}{r}}{\ln \frac{R_i}{R_o}}} \qquad (4)$$

Discussion Since the boundary conditions of the present problem are the same as those of the previous problem except that R_o and R_i are swapped, it turns out that the result is also identical except that the two radii are swapped.

9-104

Solution For modified Couette flow with two immiscible fluids we are to list the boundary conditions and then solve for both the velocity and pressure fields. Finally we are to plot the velocity profile across the channel.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the *x-y* plane.

Properties The density and viscosity of water at $T = 80^{\circ}$C are 971.8 kg/m^3 and 0.355 × 10^{-3} kg/(m·s) respectively. The density and viscosity of unused engine oil at $T = 80^{\circ}$C are 852 kg/m^3 and 32.0 × 10^{-3} kg/(m·s) respectively.

Analysis (*a*) The velocity boundary conditions come from the no-slip condition at the walls:

Boundary condition (1): At $z = 0$, $u_1 = 0$ (1)

and

Boundary condition (2): At $z = h_1 + h_2$, $u_2 = V$ (2)

At the interface we know that both the velocities and the shear stresses must match,

Boundary condition (3): At $z = h_1$, $u_1 = u_2$ (3)

and

Boundary condition (4): At $z = h_1$, $\mu_1 \dfrac{du_1}{dz} = \mu_2 \dfrac{du_2}{dz}$ (4)

The first pressure boundary condition comes from the known pressure on the bottom,

Boundary condition (5): At $z = 0$, $P = P_0$ (5)

The second pressure boundary condition comes from the fact that the pressure cannot have a discontinuity at the interface since we are ignoring surface tension,

Boundary condition (6): At $z = h_1$, $P_1 = P_2$ (6)

(*b*) We solve for the velocity field using the step-by-step procedure outlined in this chapter. However, we leave out the details because the algebra is identical to that of simple Couette flow – the only difference is in the boundary conditions. For parallel, fully developed flow in the *x* direction, *u* is the only non-zero velocity component and it is a function of *z* only. The *x* momentum equations in the two fluids reduce to

x momentum: $$\frac{d^2u_1}{dz^2} = 0 \qquad \frac{d^2u_2}{dz^2} = 0 \qquad (7)$$

We integrate both parts of Eq. 7 twice, introducing four constants of integration,

Expressions for u: $$u_1 = C_1 z + C_2 \qquad u_2 = C_3 z + C_4 \qquad (8)$$

We apply the first four boundary conditions to find these constants,

Boundary conditions (1) and (2): $$C_2 = 0 \qquad V = C_3\left(h_1 + h_2\right) + C_4$$

and

Boundary conditions (3) and (4): $$C_1 h_1 = C_3 h_1 + C_4 \qquad \mu_1 C_1 = \mu_2 C_3$$

After some algebra, we solve simultaneously for all the constants,

$$C_1 = \frac{\mu_2 V}{\mu_2 h_1 + \mu_1 h_2} \qquad C_2 = 0 \qquad C_3 = \frac{\mu_1 V}{\mu_2 h_1 + \mu_1 h_2} \qquad C_4 = V\left(\frac{\mu_2 h_1 - \mu_1 h_1}{\mu_2 h_1 + \mu_1 h_2}\right) \qquad (9)$$

And the velocity components of Eq. 8 become

$$\boxed{u_1 = \frac{\mu_2 V}{\mu_2 h_1 + \mu_1 h_2} z} \qquad (10)$$

and

$$\boxed{u_2 = \frac{\mu_1 V}{\mu_2 h_1 + \mu_1 h_2} z + V\left(\frac{\mu_2 h_1 - \mu_1 h_1}{\mu_2 h_1 + \mu_1 h_2}\right) = \frac{V}{\mu_2 h_1 + \mu_1 h_2}\left(\mu_1 \left(z - h_1\right) + \mu_2 h_1\right)} \qquad (11)$$

You should plug in the boundary conditions to verify that Eqs. 10 and 11 are correct.

(*c*) We analyze the *z* momentum equation to find the pressure. Since w = 0 everywhere, the only non-zero terms are the pressure and gravity terms. Thus we have

z momentum:
$$\frac{dP_1}{dz} = -\rho_1 g \qquad \frac{dP_2}{dz} = -\rho_2 g \qquad (12)$$

We integrate Eqs. 12 to obtain

Pressure:
$$P_1 = -\rho_1 g z + C_5 \qquad P_2 = -\rho_2 g z + C_6 \qquad (13)$$

After applying boundary conditions (5) and (6) we obtain the final expressions for the two pressures,

$$\boxed{P_1 = P_0 - \rho_1 g z} \text{ and } \boxed{P_2 = P_0 + \left(\rho_1 + \rho_2\right) g h_1 - \rho_2 g z} \qquad (14)$$

Again you can verify that the boundary conditions are satisfied by Eq. 14.

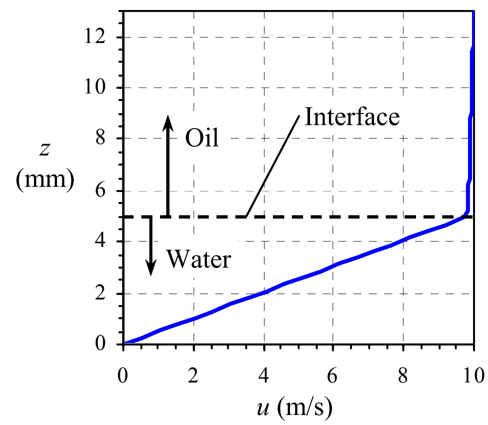


FIGURE 1
The velocity profile for Couette flow with two immiscible liquids.

(*d*) For the given fluid properties we plot the velocity profile in Fig. 1. Since the oil is so much more viscous than the water, the oil velocity is nearly constant (small slope) while the water velocity varies rapidly (large slope). At the interface the viscosity times the slope must match, so this should not be surprising.

Discussion Both velocity profiles are linear. The pressure is simply hydrostatic since *P* is a function of *z* only. The oil must be on top since it is less dense than water.

9-105

Solution We are to calculate $u(r)$ for flow inside an inclined round pipe.

Assumptions We number and list the assumptions for clarity:

1 The pipe is infinitely long in the x direction.
2 The flow is steady, i.e. any time derivative is zero.
3 This is a parallel flow (the r component of velocity, u_r, is zero).
4 The fluid is incompressible and Newtonian, and the flow is laminar.
5 The pressure is constant everywhere except for hydrostatic pressure.
6 The velocity field is axisymmetric with no swirl, implying that $u_\theta = 0$ and all derivatives with respect to θ are zero.

Analysis To obtain the velocity and pressure fields, we follow the step-by-step procedure outlined above.

Step 1 Lay out the problem and the geometry. See the problem statement.

Step 2 List assumptions and boundary conditions. We have already listed six assumptions. The first boundary condition comes from imposing the no slip condition at the pipe wall: (1) at $r = R$, $\vec{V} = 0$. The second boundary condition comes from the fact that the centerline of the pipe is an axis of symmetry: (2) at $r = 0$, $du/dr = 0$.

Step 3 Write out and simplify the differential equations. We start with the continuity equation in cylindrical coordinates, a modified version of Eq. 9-62a,

Continuity:
$$\underbrace{\cancel{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}}_{\text{Assumption 6}} + \frac{\partial u}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial x} = 0 \qquad (1)$$

Equation 1 tells us that u is not a function of x. In other words, it doesn't matter where we place our origin – the flow is the same at any x location. This can also be inferred directly from Assumption 1, which tells us that there is nothing special about any x location since the pipe is infinite in length – the flow is fully developed. Furthermore, since u is not a function of time (Assumption 2) or θ (Assumption 6), we conclude that u is at most a function of r,

Result of continuity:
$$u = u(r) \text{ only} \qquad (2)$$

We now simplify the x momentum equation as far as possible:

x momentum:
$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{Assumption 2}} + \underbrace{\cancel{u_r\frac{\partial u}{\partial r}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u}{\partial \theta}}}_{\text{Assumption 6}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\text{Continuity}}\right)$$
$$= -\underbrace{\cancel{\frac{\partial P}{\partial x}}}_{\text{Assumption 5}} + \underbrace{\rho g_x}_{\rho g \sin\alpha} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}}_{\text{Assumption 6}} + \underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{Continuity}}\right)$$

or

Result of x momentum:
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{-\rho g \sin\alpha}{\mu} \qquad (3)$$

As in previous examples the material acceleration (entire left hand side of the x momentum equation) is zero, implying that fluid particles are not accelerating at all in this flow field, and linearizing the Navier-Stokes equation. Also notice that we have replaced the partial derivative operators for the u derivatives with total derivative operators because of Eq. 2.

You can show in similar fashion that every term in the r momentum equation and in the θ momentum equation goes to zero.

Step 4 Solve the differential equations. Continuity, r momentum, and θ momentum have already been solved, and thus we are left with Eq. 3 (x momentum). After multiplying both sides by r, integrating, dividing by r, and integrating again, *Axial velocity component*: $$u = \frac{-\rho g \sin\alpha}{4\mu} r^2 + C_1 \ln r + C_2 \quad (4)$$

where C_1 and C_2 are constants of integration.

Step 5 Apply boundary conditions from Step 2 above to obtain constants C_1 and C_2. We apply boundary condition (2) first:

Boundary condition (2): $$\frac{du}{dr} = 0 + \frac{C_1}{0} = 0$$

Since $C_1/0$ is undefined (∞), the only way for du/dr to equal zero at $r = 0$ is for C_1 to equal 0. An alternative way to think of this boundary condition is to say that u must remain finite at the centerline of the pipe. Again this is possible only if constant C_1 is equal to 0.

$$C_1 = 0$$

Now we apply the first boundary condition,

Boundary condition (1):
$$u = \frac{-\rho g \sin\alpha}{4\mu} R^2 + 0 + C_2 = 0 \qquad \text{or} \qquad C_2 = \frac{\rho g \sin\alpha}{4\mu} R^2$$

Finally, Eq. 4 becomes

Final result for axial velocity: $$\boxed{u = \frac{\rho g \sin\alpha}{4\mu}\left(R^2 - r^2\right)} \quad (5)$$

The axial velocity profile is thus in the shape of a paraboloid, just as in Example 9-18.

Step 6 Verify the results. You can plug in the velocity field to verify that all the differential equations and boundary conditions are satisfied.

The volume flow rate through the pipe is found by integrating Eq. 5 through the whole cross-sectional area of the pipe,

Volume flow rate:
$$\boxed{\dot{V} = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} u dr = \frac{2\pi\rho g \sin\alpha}{4\mu}\int_{r=0}^{R}\left(R^2 - r^2\right) r dr = \frac{\pi R^4}{8\mu}\rho g \sin\alpha} \quad (6)$$

Since volume flow rate is also equal to the average axial velocity times cross-sectional area, we can easily determine the average axial velocity, V:

Average axial velocity: $$\boxed{V = \frac{\dot{V}}{A} = \frac{\frac{\pi R^4}{8\mu}\rho g \sin\alpha}{\pi R^2} = \frac{R^2}{8\mu}\rho g \sin\alpha} \quad (7)$$

Discussion There is no such thing as an “inviscid” fluid. For example, if μ were zero in this problem, the axial velocity, volume flow rate, and average velocity would all go to infinity since μ appears in the denominator of Eqs. 5 through 7.

9-106

Solution We are to generate and discuss velocity and pressure boundary conditions for the given flow problem.

Assumptions **1** The flow is steady in the mean. **2** Surface tension effects are negligibly small.

Analysis On all tank walls, $\vec{V} = 0$ since the tank walls are stationary (no-slip boundary condition). Mathematically, we write $\boldsymbol{u_r = u_\theta = u_z = 0}$ **at** $\boldsymbol{r = R_{\text{tank}}}$ (the tank side walls) and at $z = 0$ (the bottom wall of the tank). On the blade surfaces, the fluid velocity must equal that of the blades (also the no-slip condition). At any radial location r the velocity of the blade surface is $\vec{V}_{\text{blade}} = r\omega\vec{e}_\theta$. In other words $\boldsymbol{u_\theta = r\omega}$ **at the blade surfaces**. Since the blades do not move at all in the radial or vertical directions, $\boldsymbol{u_r = u_z = 0}$ **along the blade surfaces**. Finally, **at the free surface** $\boldsymbol{P = P_{\text{atm}}}$ since the free surface is exposed to atmospheric air. In addition, the vertical component of velocity $\boldsymbol{u_z}$ **must equal zero at the free surface**. We note that the other two velocity components (u_r and u_θ) may be non-zero at the free surface, but the shear stress in the horizontal plane of the free surface must be zero (negligible shear due to the air). Mathematically, $\boldsymbol{\partial u_r/\partial z = \partial u_\theta/\partial z = 0}$ **at the free surface**.

Discussion The no-slip condition requires that $u_\theta = r\omega$ everywhere on the blade surface, regardless of the geometry of the blades.

9-107

Solution We are to generate and discuss velocity and pressure boundary conditions for the stirrer flow problem from a rotating frame of reference.

Assumptions **1** The flow is steady in the mean. **2** Surface tension effects are negligibly small.

Analysis On all tank walls, $\vec{V}_{\text{tank}} = 0$ from a stationary frame of reference since the tank walls are stationary (no-slip boundary condition). From the *rotating* frame of reference however, the tank walls are rotating in the opposite direction of ω. Mathematically, we write $\boldsymbol{u_r = u_z = 0}$ **and** $\boldsymbol{u_\theta = -R_{\text{tank}}\omega}$ **at** $\boldsymbol{r = R_{\text{tank}}}$ (the tank side walls). **At the bottom wall of the tank we write** $\boldsymbol{u_r = u_z = 0}$ **and** $\boldsymbol{u_\theta = -r\omega}$ **at** $\boldsymbol{z = 0}$. On the blade surfaces, the fluid velocity must equal that of the blades (also the no-slip condition). Since the blades are stationary in this rotating frame of reference, $\boldsymbol{u_r = u_z = u_\theta = 0}$ **at the blade surfaces**. Finally, **at the free surface** $\boldsymbol{P = P_{\text{atm}}}$ since the free surface is exposed to atmospheric air. In addition, the vertical component of velocity u_z must equal zero at the free surface. We note that the other two velocity components (u_r and u_θ) may be non-zero at the free surface, but the shear stress in the horizontal plane of the free surface must be zero (negligible shear due to the air). Mathematically, $\boldsymbol{\partial u_r/\partial z = \partial u_\theta/\partial z = 0}$ **at the free surface**.

Discussion In this problem the free surface boundary conditions are independent of frame of reference.

Review Problems

9-108C

Solution We are to list the six steps used to solve fluid flow problem with the continuity and Navier-Stokes equations, for the case in which the fluid is incompressible and has constant properties.

Analysis The steps are listed below:

Step 1 Lay out the problem and the geometry. Identify all relevant dimensions and parameters.
Step 2 List all appropriate assumptions, approximations, simplifications, and boundary conditions.
Step 3 Write out and simplify the differential equations (continuity and the required components of Navier-Stokes) as much as possible.
Step 4 Solve (integrate) the differential equations. This leads to one or more constants of integration.
Step 5 Apply boundary conditions to obtain values for the constants of integration.
Step 6 Verify the results by checking that the flow field meets all the specifications and boundary conditions.

Discussion These steps are not always followed in the same order. For example, in CFD applications the boundary conditions are applied *before* the equations are integrated.

9-109C

Solution We are to name each equation, and then discuss its restrictions and its physical meaning.

Analysis (*a*) This is the **continuity equation**. The form given here is valid for any fluid. It describes conservation of mass in a fluid flow.

(*b*) This is **Cauchy's equation**. The form given here is valid for any fluid. It describes conservation of linear momentum in a fluid flow.

(*c*) This is the **Navier-Stokes equation**. The form given here is valid for a specific type of fluid, namely an incompressible Newtonian fluid. The equation describes conservation of linear momentum in a fluid flow.

Discussion It is important that you be able to recognize these notable equations of fluid mechanics.

9-110C

Solution We are to discuss the connection(s) between the incompressible flow approximation and the constant temperature approximation.

Analysis For an incompressible flow, the density is assumed to be constant. In addition, the incompressible flow approximation usually implies that *all* fluid properties (viscosity, thermal conductivity, etc.) are constant as well. These assumptions go hand in hand because a flow with constant density implies a flow with little or no temperature changes and no buoyancy effects. Since viscosity is a strong function of temperature but generally a weak function of pressure, the fluid's viscosity is approximately constant whenever temperature is constant. When dealing with incompressible fluid flows, pressure variable P is interpreted as the mechanical pressure P_m, and we don't need an equation of state. In effect, the equation of state is replaced by the assumption of constant density and constant temperature.

Discussion Mechanical pressure P_m is determined by the flow field, not by thermodynamics.

9-111C

Solution

(*a*) **True**: The unknowns for an incompressible flow problem with constant fluid properties are pressure and the three components of velocity. Density and viscosity are constants and are therefore not unknowns.

(*b*) **False**: The unknowns for a compressible flow problem are pressure, the three components of velocity, and the density. However, density is a thermodynamic function of pressure and temperature. Hence, temperature appears as an additional unknown, as does some kind of equation of state. In summary, there are actually at least 6 unknowns (P, u, v, w, ρ, and T). We therefore need 6 equations (continuity, 3 components of Navier-Stokes, equation of state, and energy). In addition, fluid properties such as viscosity may change as well, and we need either more equations or some kind of look-up table for these properties.

(*c*) **False**: Cauchy's equation contains additional unknowns – the components of the stress tensor, which must be written in terms of the velocity and pressure fields through some kind of constitutive equation.

(d) **True**: For an incompressible flow problem involving a Newtonian fluid, there are only 4 unknowns (P, u, v, and w). We therefore need only 4 equations (continuity and 3 components of Navier-Stokes).

9-112C

Solution We are to discuss the relationship between volumetric strain rate and the continuity equation.

Analysis Volumetric strain rate is defined as the rate of increase of volume of a fluid element per unit volume. In a compressible flow field, the volume of a fluid particle may increase or decrease as it moves along in the flow, but its mass must remain constant. (This is a fundamental statement of conservation of mass of a system, since the fluid particle can be thought of as an infinitesimal system.) Mathematically it turns out that **volumetric strain rate is the sum of the three normal strain rates, and is identically zero for incompressible flow** (density cannot change, and hence volume cannot change). The continuity equation is based on the same fundamental principle of mass conservation. It is a differential form of the equation of conservation of mass. Its incompressible form also shows that the sum of the three normal strain rates must be zero. On the other hand, if the density is *not* constant, the sum of the three normal strain rates is not zero, but is still equal to the volumetric strain rate, which is also non-zero.

Discussion Volumetric strain rate is derived and discussed in Chap. 4 as a kinematic property.

9-113

Solution For a given geometry and set of boundary conditions, we are to calculate the velocity and pressure fields, and plot the velocity profile.

Assumptions The assumptions are identical to those of Example 9-17. We do not list them here.

Analysis We obtain the velocity and pressure fields by following the step-by-step procedure for differential fluid flow solutions. Everything is identical to Example 9-17 except for the boundary condition at the wall. Boundary condition (1), the no-slip condition, becomes: at $x = 0$, $u = v = 0$. $w = V$. Steps 1 through 4 are otherwise identical, and the result is

Result of integration of z momentum: $$w = \frac{\rho g}{2\mu}x^2 + C_1 x + C_2 \qquad (1)$$

We continue, beginning with Step 5:

Step 5 We apply boundary conditions (1) and (2) from Step 2 to obtain constants C_1 and C_2,

Boundary condition (1): $$w = 0 + 0 + C_2 = V \qquad C_2 = V$$

and

Boundary condition (2): $$\left.\frac{dw}{dx}\right)_{x=h} = \frac{\rho g}{\mu}h + C_1 = 0 \qquad C_1 = -\frac{\rho g h}{\mu}$$

Finally, Eq. 1 becomes

Result: $$\boxed{w = \frac{\rho g}{2\mu}x^2 - \frac{\rho g}{\mu}hx + V = \frac{\rho g x}{2\mu}\left(x - 2h\right) + V} \qquad (2)$$

Since $x < h$ in the film, the first term in Eq. 2 is negative, but the second term is positive. Depending on the relative magnitude of the terms, part or all of the vertical velocity may be positive. The pressure field is still $P = P_{atm}$ everywhere.

Step 6 Verify the results. You can plug in the velocity field to verify that all the differential equations and boundary conditions are satisfied.

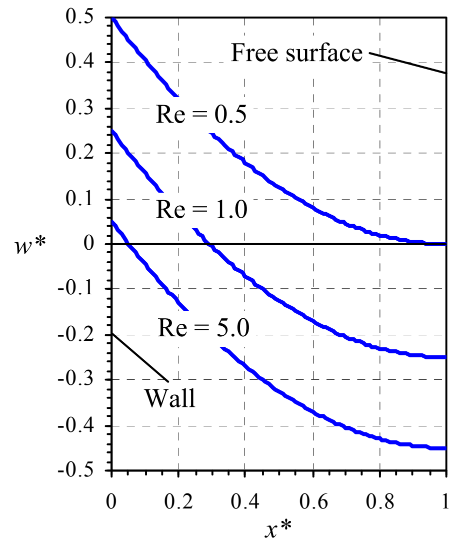


FIGURE 1
Velocity profiles for an oil film falling down a moving vertical wall. For all three Reynolds numbers, Fr = 0.5.

We nondimensionalize Eq. 2 by inspection: we let $x^* = x/h$ and $w^* = w\mu/(\rho g h^2)$. Eq. 2 becomes

Nondimensional velocity profile: $$w^* = \frac{x^*}{2}\left(x^* - 2\right) + \frac{V\mu}{\rho g h^2} \qquad (3)$$

We verify by inspection that when $V = 0$, Eq. 3 reduces to the velocity profile of Example 9-17. After some algebra we see that Eq. 3 can be re-written as

Final nondimensional velocity profile: $$w^* = \frac{x^*}{2}\left(x^* - 2\right) + \frac{\mathrm{Fr}^2}{\mathrm{Re}} \qquad (4)$$

where Froude number $\mathrm{Fr} = V/\sqrt{gh}$ and Reynolds number $\mathrm{Re} = \rho V h/\mu$. We plot the nondimensional velocity field in Fig. 1 for Fr = 0.5 and Re = 0.5, 1.0, and 5.0.

Discussion Notice that the velocity profile has zero slope at the free surface regardless of the values of Fr and Re. For large enough V, the net mass flow rate is upward rather than downward.

9-114

Solution We are to calculate the volume flow rate per unit width of oil falling down a moving vertical wall, and then calculate the wall speed such that the net volume flow rate of oil is zero.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The wall is infinitely wide and very long so that all of the parallel flow, fully developed approximations of the previous problem hold.

Analysis We calculate the volume flow rate per unit width by integration of the velocity:

Volume flow rate per unit depth:

$$\frac{\dot{V}}{L} = \int_0^h w dx = \int_0^h \left[\frac{\rho g x}{2\mu}(x - 2h) + V \right] dx = \boxed{Vh - \frac{\rho g h^3}{3\mu}} \qquad (1)$$

The volume flow rate is zero when the two terms in Eq. 1 cancel,

Zero volume flow rate:

$$\frac{\dot{V}}{L} = 0 \text{ when } Vh = \frac{\rho g h^3}{3\mu} \text{ or } \boxed{V = \frac{\rho g h^2}{3\mu}} \qquad (2)$$

For an oil film of thickness 5.65 mm with ρ = 888 kg/m^3 and μ = 0.80 kg/(m·s), we calculate V using Eq. 2,

Result for V:

$$V = \frac{\rho g h^2}{3\mu} = \frac{(888\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.00412\text{ m})^2}{3(0.801\text{ kg/m}\cdot\text{s})} = 0.061535\text{ m/s} \cong \mathbf{0.0615\text{ m/s}} \qquad (3)$$

Discussion For any V greater than the value calculated in Eq. 3, the net oil flow is up, while for V less than this value, the net oil flow is down. Since viscosity is in the denominator of Eq. 2, a low viscosity liquid (like water) would require a very large vertical velocity in order to achieve a net upward flow of the liquid.

9-115E

Solution For a given axial velocity component in an axisymmetric flow field, we are to validate the incompressible approximation, generate the radial velocity component, generate an expression for the stream function, and then plot some streamlines and design the shape of the contraction.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is axisymmetric implying that $u_\theta = 0$ and there is no variation in the θ direction.

Properties At room temperature and pressure, the speed of sound in air is about 1130 ft/s.

Analysis (*a*) The maximum speed occurs in the test section, where the Mach number is

Mach number:
$$\text{Ma} = \frac{u_{z,L}}{c} = \frac{120\ \frac{\text{ft}}{\text{s}}}{1130\ \frac{\text{ft}}{\text{s}}} = 0.106 \qquad (1)$$

Since Ma is much less than 0.3, *the incompressible flow approximation is reasonable*.

(*b*) Between $z = 0$ and $z = L$, the axial velocity component is given by

Axial velocity component:
$$u_z = u_{z,0} + \frac{u_{z,L} - u_{z,0}}{L} z \qquad (2)$$

We use the incompressible continuity equation in cylindrical coordinates, simplified as follows for axisymmetric flow,

Incompressible axisymmetric continuity equation:
$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial(u_z)}{\partial z} = 0$$

After rearranging,

$$\frac{\partial(ru_r)}{\partial r} = -r\frac{\partial(u_z)}{\partial z} = -r\frac{u_{z,L} - u_{z,0}}{L} \qquad (3)$$

We integrate Eq. 3 with respect to r,

$$ru_r = -\frac{r^2}{2}\frac{u_{z,L} - u_{z,0}}{L} + f(z) \qquad (4)$$

Notice that since we performed a *partial* integration with respect to r, we add a function of the other variable z rather than simply a constant of integration. We divide all terms in Eq. 4 by r and recognize that the term with $f(z)$ will go to infinity at the centerline of the contraction ($r = 0$) unless $f(z) = 0$. Our final expression for u_r is thus

Radial velocity component:
$$\boxed{u_r = -\frac{r}{2}\frac{u_{z,L} - u_{z,0}}{L}} \qquad (5)$$

(*c*) The algebra for generating the stream function is identical to that of Problem 9-61 except for a change in notation. The result is thus

Stream function:
$$\boxed{\psi = \frac{r^2}{2}\left(u_{z,0} + \frac{u_{z,L} - u_{z,0}}{L}z\right) + \text{constant}} \qquad (6)$$

The constant can be anything. We set it to zero for simplicity.

(*d*) First we calculate the axial speed at the entrance to the contraction. By conservation of mass,

$$u_{z,0}A_0 = u_{z,L}A_L \qquad \text{or} \qquad u_{z,0}\frac{\pi D_0^{\,2}}{4} = u_{z,L}\frac{\pi D_L^{\,2}}{4}$$

from which

$$u_{z,0} = u_{z,L}\frac{D_L^{\,2}}{D_0^{\,2}} = 120\frac{\text{ft}}{\text{s}} \times \frac{(1.5\text{ ft})^2}{(5.0\text{ ft})^2} = 10.8\frac{\text{ft}}{\text{s}}$$

We solve Eq. 6 for r as a function of z and plot several streamlines in Fig. 1,

Streamlines:
$$r = \pm\sqrt{\frac{2\psi}{u_{z,0} + \dfrac{u_{z,L} - u_{z,0}}{L}z}} \qquad (7)$$

At the entrance of the contraction (z = 0), the wall is at $r = D_0/2$ = 2.5 ft. Eq. 6 yields ψ_{wall} = 33.75 ft^3/s for the streamline that passes through this point. This streamline thus represents the shape of the nozzle wall, and we have designed the nozzle shape.

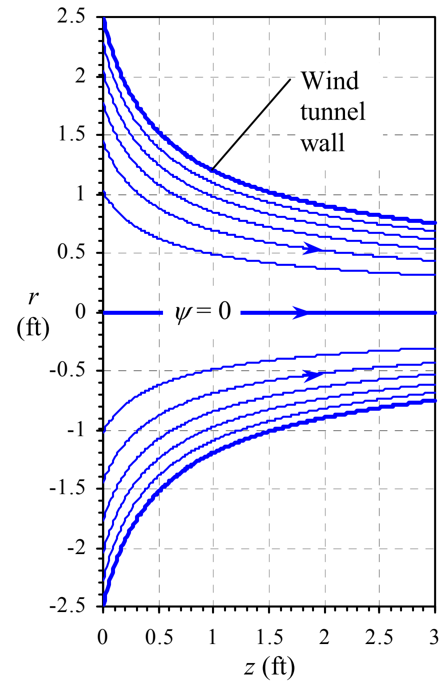


FIGURE 1
Streamlines for flow through an axisymmetric wind tunnel contraction.

Discussion Since the boundary layers along the walls of the contraction are very small, the assumption about negligible friction effects is reasonable. This contraction shape should deliver the desired axial flow speed quite nicely.

9-116

Solution We are to determine a relationship between constants *a*, *b*, *c*, *d*, and *e* that ensures incompressibility, and we are to determine the primary dimensions of each constant.

Assumptions **1** The flow is steady. **2** The flow is incompressible (under certain restraints to be determined).

Analysis We plug the velocity components into the incompressible continuity equation,

Condition for incompressibility:

$$\underbrace{\frac{\partial u}{\partial x}}_{az^2}+\underbrace{\frac{\partial v}{\partial y}}_{cxz}+\underbrace{\frac{\partial w}{\partial z}}_{3dz^2+2exz}=0 \qquad az^2+cxz+3dz^2+2exz=0 \qquad (1)$$

To guarantee incompressibility, the above equation must be satisfied everywhere. We equate similar terms to obtain the following relationships:

Conditions for incompressibility:

$$\boxed{a=-3d \qquad c=-2e} \qquad (2)$$

The units are found by observing that each component of the velocity field must be dimensionally homogeneous – each term must have dimensions of velocity. We examine each term:

$$\left\{axz^2\right\}=\left\{a\times \mathrm{L}^3\right\}=\left\{\frac{\mathrm{L}}{\mathrm{t}}\right\} \qquad \boxed{\left\{a\right\}=\left\{\frac{1}{\mathrm{L}^2\mathrm{t}}\right\}}$$

$$\left\{by\right\}=\left\{b\times \mathrm{L}\right\}=\left\{\frac{\mathrm{L}}{\mathrm{t}}\right\} \qquad \boxed{\left\{b\right\}=\left\{\frac{1}{\mathrm{t}}\right\}}$$

$$\left\{cxyz\right\}=\left\{c\times \mathrm{L}^3\right\}=\left\{\frac{\mathrm{L}}{\mathrm{t}}\right\} \qquad \boxed{\left\{c\right\}=\left\{\frac{1}{\mathrm{L}^2\mathrm{t}}\right\}}$$

$$\left\{dz^3\right\}=\left\{d\times \mathrm{L}^3\right\}=\left\{\frac{\mathrm{L}}{\mathrm{t}}\right\} \qquad \boxed{\left\{d\right\}=\left\{\frac{1}{\mathrm{L}^2\mathrm{t}}\right\}}$$

$$\left\{exz^2\right\}=\left\{e\times \mathrm{L}^3\right\}=\left\{\frac{\mathrm{L}}{\mathrm{t}}\right\} \qquad \boxed{\left\{e\right\}=\left\{\frac{1}{\mathrm{L}^2\mathrm{t}}\right\}}$$

Discussion If Eq. 2 were not satisfied, the given velocity field might still represent a valid flow field, but density would have to vary with location in the flow field – in other words the flow would be *compressible*.

9-117

Solution We are to simplify the incompressible Navier-Stokes equation for the case of rigid body motion with arbitrary acceleration.

Analysis We begin with the vector form of the incompressible Navier-Stokes equation,

Incompressible Navier-Stokes equation:
$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \qquad (1)$$

In rigid body motion, $\vec{V}$ is not zero, but since the liquid moves as a solid body there is no *relative* motion between fluid particles. Thus the viscous term in Eq. 1 disappears. (Fluid particles do not rub against each other or shear against each other in any way, so the viscous term must vanish.) The material acceleration term $D\vec{V}/Dt$ is the acceleration following a fluid particle; hence it is identical to the imposed acceleration $\vec{a}$. Finally, $\vec{g} = -g\vec{k}$. Thus Eq. 1 reduces to

Equation for rigid body acceleration:
$$\boxed{\vec{\nabla}P + \rho g\vec{k} = -\rho\vec{a}} \qquad (2)$$

Discussion You can verify that Eq. 2 agrees with the rigid body acceleration equation of Chap. 3.

9-118

Solution We are to simplify the incompressible Navier-Stokes equation for the case of hydrostatics.

Analysis We begin with the vector form of the incompressible Navier-Stokes equation,

Incompressible Navier-Stokes equation:
$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \qquad (1)$$

In hydrostatics, $\vec{V} = 0$ everywhere (no flow). Thus the first and last terms in Eq. 1 disappear. In addition, $\vec{g} = -g\vec{k}$. Thus Eq. 1 reduces to

Hydrostatics equation:
$$\boxed{\vec{\nabla}P = -\rho g\vec{k}} \qquad (2)$$

Discussion We verify from Eq. 2 that pressure does not change horizontally, but increases downward.

9-119

Solution We are to specify boundary conditions in terms of stream function.

Assumptions **1** The flow is steady. **2** The flow is incompressible **3** The flow is two-dimensional.

Analysis (*a*) For 2-D incompressible flow the difference in the value of the stream function between two streamlines is equal to the volume flow rate per unit width between the two streamlines. Since the entire flow is confined between the lower and upper channel walls, we know that stream function ψ must be constant along the upper wall. We calculate ψ on the upper channel wall as follows:

$$V_1 = \frac{\dot{V}}{H_1 W} = \frac{1}{H_1}\frac{\dot{V}}{W} = \frac{1}{H_1}\left(\psi_{\text{upper}} - \psi_{\text{lower}}\right) \quad (1)$$

from which

ψ_{upper}: $$\psi_{\text{upper}} = \psi_{\text{lower}} + H_1 V = 0 + (0.12\text{ m})(18.5\text{ m/s}) = \mathbf{2.22\ m^2/s} \quad (2)$$

(*b*) Since the inlet flow is uniform, ψ must increase linearly from ψ_{lower} to ψ_{upper} along the left edge of the computational domain. In equation form,

ψ_{left}: $$\psi_{\text{left}} = \psi_{\text{lower}} + \frac{\left(\psi_{\text{upper}} - \psi_{\text{lower}}\right)}{H_1} y = \frac{2.22\text{ m}^2\text{/s}}{0.12\text{ m}} y = \boxed{(18.5\text{ m/s})\,y} \quad (3)$$

We notice that Eq. 3 could have been obtained directly from $u = V_1 = \partial\psi/\partial y$.

(*c*) We have some options for the right edge of the computational domain. If that boundary is far enough away that it does not adversely affect the flow near the sudden contraction, we might specify a uniform velocity distribution along the right edge, similar to Eq. 3 above, but with a higher velocity determined by conservation of mass,

Average outlet speed: $$V_2 = V_1 \frac{H_1}{H_2} = (18.5\text{ m/s})\frac{0.12\text{ m}}{0.046\text{ m}} = 48.26\text{ m/s}$$

In other words, we would specify

ψ_{right}: $$\boxed{\psi_{\text{right}} = (48.26\text{ m/s})\,y} \quad (4)$$

Eq. 4 is not a very good boundary condition because we know that viscous effects will surely slow down the flow near the walls – the velocity profile at the outlet will *not* be uniform.

A much better boundary condition (if the code permits it) is to specify that ψ not change with x along the right edge of the domain. Mathematically, we would specify

ψ_{right}: $$\frac{\partial \psi_{\text{right}}}{\partial x} = 0 \quad (5)$$

You can see from the definition of ψ that Eq. 5 is identical to forcing velocity component v to be zero at the outlet. In other words, we specify that the flow at the outlet is *parallel*.

A third option would be to locate the right edge very far downstream so that the flow there is fully developed channel flow, for which we can specify the stream function as a function of y along the edge. ψ can be obtained from Problem 9-43.

Discussion CFD and boundary conditions are discussed in detail in Chap. 15.

9-120

Solution For each equation we are to tell whether it is linear or nonlinear and explain.

Analysis (*a*) The incompressible continuity equation is

The incompressible continuity equation: $$\vec{\nabla}\cdot\vec{V}=0 \qquad (1)$$

This equation is linear. There are no nonlinear terms.

(*b*) The compressible continuity equation is

The compressible continuity equation: $$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\left(\rho\vec{V}\right)=0 \qquad (2)$$

This equation is nonlinear. The second term has a product of two variables, ρ and $\vec{V}$ – this is what makes the equation nonlinear.

(*c*) The incompressible Navier-Stokes equation is

The incompressible Navier-Stokes equation: $$\rho\frac{D\vec{V}}{Dt}=-\vec{\nabla}P+\rho\vec{g}+\mu\nabla^2\vec{V} \qquad (3)$$

This equation is nonlinear. The material acceleration term on the left can be written as

The incompressible Navier-Stokes equation: $$\frac{D\vec{V}}{Dt}=\underbrace{\frac{\partial\vec{V}}{\partial t}}_{\text{Unsteady or local part}}+\underbrace{\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}}_{\text{Advective (or convective) part}} \qquad (4)$$

The advective part of Eq. 4 contains products of variable $\vec{V}$ and derivatives of variable $\vec{V}$ – this is what makes the equation nonlinear.

Discussion Density is treated as a constant in Eq. 3, and does not affect the nonlinearity of the equation. For *compressible* flow however, variable density causes the nonlinearity.

9-121

Solution We are to sketch some streamlines for boundary layer flow.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis We can offer only quantitative sketches of the streamlines. Since there is no flow reversal, we can be sure that **$\delta(x)$ is not a streamline**. In fact, streamlines must *cross* $\delta(x)$. Furthermore, at any given *y* location above the plate the fluid speed decreases as the boundary layer grows downstream. Hence, *the streamlines must diverge*. The bottom line is that the streamlines veer slightly upward away from the wall to compensate for the loss of speed in the boundary layer. Streamlines are sketched in Fig. 1.

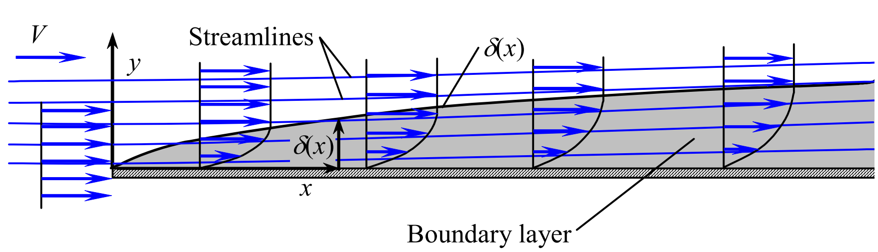


FIGURE 1
Streamlines above and within a flat plate boundary layer; since streamlines cross the curve $\delta(x)$, $\delta(x)$ cannot itself be a streamline of the flow. Furthermore, streamlines within the boundary layer veer up because of decreasing speeds within the boundary layer.

Discussion As the boundary layer grows in thickness, more and more streamlines end up inside the boundary layer.

9-122

Solution We are to define a ψ that satisfies the continuity equation, and increases in the positive z direction when the flow is from right to left in the x-z plane.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-z plane.

Analysis We propose the following stream function,

Stream function:
$$\boxed{u = -\frac{\partial \psi}{\partial z} \qquad w = \frac{\partial \psi}{\partial x}} \tag{1}$$

We verify that the continuity equation is satisfied by Eq. 1,

Steady, incompressible, 2-D continuity equation:
$$\underbrace{\frac{\partial u}{\partial x}}_{-\frac{\partial^2 \psi}{\partial x \partial z}} + \underbrace{\frac{\partial w}{\partial z}}_{\frac{\partial^2 \psi}{\partial z \partial x}} = 0 \tag{2}$$

The only restriction is that ψ must be a smooth function of x and z. We check if we picked the proper signs by examining freestream flow from right to left in the x-z plane:

Freestream flow:
$$u = -U \qquad w = 0 \qquad \psi = Uz + C \tag{3}$$

where U is a positive constant and C is an arbitrary constant. Thus we verify that as z increases, ψ increases, and the flow is from right to left as desired.

Discussion If we had defined ψ with the opposite signs of Eq. 1, the flow would be from left to right as ψ increases.

9-123

Solution We are to analyze this problem two ways: with the exact (differential) technique, and with dimensional analysis, and we are to compare the results.

Assumptions **1** The flow is steady. **2** The flow is incompressible, Newtonian, laminar, parallel, and fully developed ($u = u(y)$ only, where x is in the direction of motion and y is normal to the direction of motion). **3** We ignore aerodynamic drag on the block.

Analysis (*a*) We draw a free-body diagram of the block in Fig. 1 and sum all the forces acting on it. There are only two forces in the x direction: the x component of weight $W\sin\alpha$ and the force τA due to viscous shear at the bottom surface of the block. Since the block slides at constant speed, these two forces must balance.

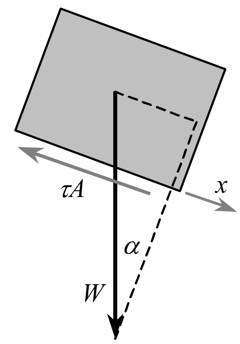


FIGURE 1
Free-body diagram of the block.

Force balance:
$$W \sin\alpha = \tau A = \frac{\mu V A}{h} \qquad (1)$$

where we have used the exact analytical expression for the shear stress for Couette flow, namely $\tau = \mu(du/dy) = \mu V/h$. Solving for h,

Exact solution for h:
$$\boxed{h = \frac{\mu V A}{W \sin\alpha}} \qquad (2)$$

(*b*) We perform a dimensional analysis leaving out many of the details. There are 6 parameters in the problem: h as a function of V, A, W, α, and μ. There are three primary dimensions represented in the problem, namely m, L, and t. Thus we expect 6-3 = 3 Πs. We choose three repeating variables, V, A, and W. The Πs are

Dimensionless parameters:
$$\Pi_1 = \frac{h}{\sqrt{A}} \qquad \Pi_2 = \frac{\mu V \sqrt{A}}{W} \qquad \Pi_3 = \alpha$$

The dimensionless relationship is

Result of dimensional analysis:
$$\boxed{\frac{h}{\sqrt{A}} = f\left(\frac{\mu V \sqrt{A}}{W}, \alpha\right)} \qquad (3)$$

To put the Πs of Eq. 3 into the form of Eq. 2 we do the following:

Relationship between Πs:
$$\Pi_1 = \frac{\Pi_2}{\sin\Pi_3} \quad \rightarrow \quad \frac{h}{\sqrt{A}} = \frac{\mu V \sqrt{A}}{W \sin\alpha} \quad \rightarrow \quad \boxed{h = \frac{\mu V A}{W \sin\alpha}} \qquad (4)$$

Thus we see that dimensional analysis is indeed consistent with the exact solution. Of course, we could not know the relationship of Eq. 4 by dimensional reasoning alone.

Discussion The agreement between Parts (*a*) and (*b*) is satisfying and emphasizes two different approaches to the same engineering problem.

9-124

Solution We are to write Poisson's equation in standard form and discuss its similarities and differences compared to Laplace's equation.

Analysis Poisson's equation in standard form is

Poisson's equation: $$\boxed{\nabla^2\phi = s} \qquad (1)$$

where ϕ is a dependent variable that is a function of space, ∇^2 is the Laplacian operator, and s is the right hand side of the equation, which may be a function of space, but cannot be a function of ϕ itself. Poisson's equation is similar to Laplace's equation in that the left hand sides are identical. The difference is that Poisson's equation has a non-zero right hand side whereas the right hand side of Laplace's equation is zero. Note: Poisson's equation reduces to Laplace's equation if $s = 0$.

Discussion We discuss Poisson's equation briefly in this chapter in relation to pressure correction algorithms used by CFD codes.

9-125

Solution We are to analyze this problem three ways: with the control volume technique, with the differential technique, and with dimensional analysis, and we are to compare the results.

Assumptions **1** The flow is steady. **2** The flow is axisymmetric, incompressible, Newtonian, laminar, parallel, and fully developed ($u = u(r)$ only).

Analysis (a) We use the head form of the energy equation from point 1 to point 2. Since there are no pumps, turbines, or minor losses the energy equation reduces to

Energy equation:
$$\frac{P_1}{\rho g}+\alpha_1\frac{V_1^2}{2g}+z_1=\frac{P_2}{\rho g}+\alpha_2\frac{V_2^2}{2g}+z_2+h_f \tag{1}$$

The pressure terms cancel since $P_1 = P_2 = P_{atm}$. The velocity terms cancel since the flow is fully developed. Upon substitution of the major head loss equation we have

Reduced energy equation:
$$\Delta z = z_1 - z_2 = h_f = f\frac{L}{D}\frac{V^2}{2g} \tag{2}$$

But for fully developed laminar pipe flow we know from Chap. 6 that the Darcy friction factor f = 64/Re. Thus Eq. 2 becomes

$$\Delta z = \frac{64}{\text{Re}}\frac{L}{D}\frac{V^2}{2g}=\frac{64\mu}{\rho VD}\frac{L}{D}\frac{V^2}{2g}=\frac{32\mu LV}{\rho D^2 g}$$

from which we can solve for average velocity V through the pipe,

V from control volume analysis:
$$\boxed{V=\frac{\rho g D^2\Delta z}{32\mu L}} \tag{3}$$

(b) An exact analysis of this flow was performed in Problem 9-100. We refer to the solution of that problem and do not show the details here. The average velocity through the pipe was found to be

$$V=\frac{R^2}{8\mu}\rho g\sin\alpha$$

But $R = D/2$, and from the figure provided in the problem statement we see that $\sin\alpha = \Delta z/L$. Thus, our result is

V from differential analysis:
$$\boxed{V=\frac{\rho g D^2\Delta z}{32\mu L}} \tag{4}$$

The agreement with the result of Part (a) is exact.

(c) Finally we perform a dimensional analysis. We leave out the details, providing only a summary here; this is a good review of the material of Chap. 7. There are 7 parameters in the problem: V as a function of ρ, g, D, Δz, μ, and L. There are three primary dimensions represented in the problem, namely m, L, and t. Thus we expect 7-3 = 4 Πs. We choose three repeating variables, ρ, g, and D. The Πs are

Dimensionless parameters:
$$\Pi_1=\frac{V}{\sqrt{gD}}\qquad \Pi_2=\frac{\rho D\sqrt{gD}}{\mu}\qquad \Pi_3=\frac{\Delta z}{D}\qquad \Pi_4=\frac{L}{D}$$

The first Π is a Froude number and the second Π is a Reynolds number. The dimensionless relationship is

Result of dimensional analysis:
$$\boxed{\frac{V}{\sqrt{gD}}=f\left(\frac{\rho D\sqrt{gD}}{\mu},\frac{\Delta z}{D},\frac{L}{D}\right)} \tag{5}$$

To put the Πs of Eq. 5 into the form of Eq. 4 we do the following:

Relationship between Πs: $$\Pi_1 = \frac{\Pi_2\Pi_3}{32\Pi_4} \quad \rightarrow \quad \frac{V}{\sqrt{gD}} = \frac{\rho D\sqrt{gD}}{32\mu}\frac{\Delta z}{D}\frac{D}{L} \quad \rightarrow \quad \boxed{V = \frac{\rho g D^2 \Delta z}{32\mu L}} \tag{6}$$

Thus we see that dimensional analysis is indeed consistent with the exact solution. Of course, we could not know the relationship of Eq. 6 by dimensional reasoning alone.

Discussion The agreement between Parts (*a*), (*b*), and (*c*) is satisfying and emphasizes three different approaches to the same engineering problem.

9-126

Solution We are to determine the primary dimensions of ψ, nondimensionalize Eq. 1, and then plot several nondimensional streamlines for this flow field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y or r-θ plane.

Analysis

(a) There are several ways to calculate the primary dimensions of ψ. First, from Eq. 1 we see that

Dimensions of stream function:
$$\boxed{\{\psi\} = \left\{\frac{\dot{V}}{2\pi L}\right\} = \left\{\frac{\mathrm{L^3 t^{-1}}}{\mathrm{L}}\right\} = \left\{\frac{\mathrm{L^2}}{\mathrm{t}}\right\}} \qquad (2)$$

We could also use the definition of ψ. Since velocities are obtained by spatial derivatives of ψ, ψ must have an additional length dimension in the numerator compared to the dimensions of velocity. This reasoning also yields $\{\psi\} = \{\mathrm{L^2/t}\}$.

(b) The nondimensional form of the stream function is straightforward. Eq. 1 becomes

Nondimensional stream function:
$$\psi^* = -\arctan\frac{\sin 2\theta}{\cos 2\theta + \dfrac{1}{r^{*2}}} \qquad (3)$$

(c) We solve Eq. 3 for r^*,

Equation for nondimensional streamlines:
$$\boxed{r^* = \pm\sqrt{\frac{\tan(-\psi^*)}{\sin 2\theta - \cos 2\theta \tan(-\psi^*)}}} \qquad (4)$$

We pick the positive root to avoid negative radii. We plot several streamlines in the desired range in Fig. 1. The range of ψ^* is 0 on the positive x axis to $-\pi$ on the positive y axis to -2π on the negative x axis.

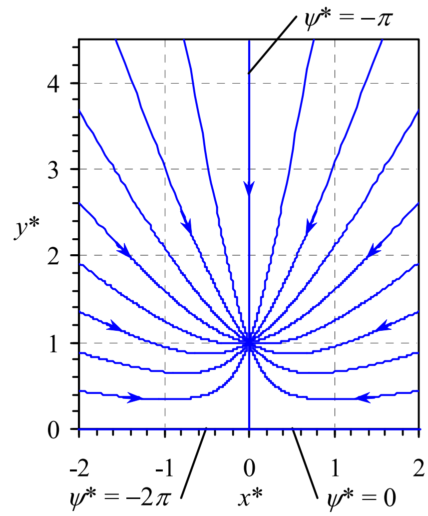


FIGURE 1
Nondimensional streamlines for flow into a vacuum cleaner attachment; ψ^* is incremented uniformly from 2π (negative x axis) to 0 (positive x axis).

Discussion The point (x = 0, y = b) is a singularity point with infinite velocity.

9-127

Solution Two examples in Chapter 9 developed a velocity profile and flow rate equation by making assumptions in pipe flow where one of these examples yielded the Poiseuille flow equation. In this problem, we assume the fluid, blood here, is not Newtonian but a Bingham Plastic fluid. The velocity profile and flow rate are to be determined. In addition, the velocity profile is to be plotted along with Newtonian and Pseudoplastic fluids.

Assumptions: **1** The flow is steady and incompressible. **2** The flow occurs within a rigid, circular pipe with uniformity and is axisymmetric.

Analysis Following the Power Law example in Chap. 9, we want to derive the velocity profile and flow rate using the shear stress equation provided for a Bingham Plastic model as noted below,

$$\tau = -\mu\frac{du}{dr} + \tau_y \text{ and recall that } \tau = \frac{r}{2}\frac{dP}{dz}$$

So, if we substitute for τ and rearrange to get

$$\frac{du}{dr} = -\frac{r}{2\mu}\frac{dP}{dz} + \frac{\tau_y}{\mu}$$

$$du = -\frac{1}{2\mu}\frac{dP}{dz}r\,dr + \frac{\tau_y}{\mu}dr$$

Integrate,

$$u = -\frac{1}{4\mu}\frac{dP}{dz}r^2 + \frac{\tau_y}{\mu}r + C$$

where C is a constant of integration.

Apply the boundary condition, $at\ r = R,\ u = 0$, which yields

$$C = \frac{1}{4\mu}\frac{dP}{dz}R^2 + \frac{\tau_y}{\mu}R$$

We then substitute back in to arrive at a velocity equation

$$u = \frac{1}{4\mu}\frac{dP}{dz}(R^2 - r^2) - \frac{\tau_y}{\mu}(R - r)$$

However, this is not complete. Because there is a yield stress, we will have to treat the solution a little differently and need to break up the flow into 2 sections. For flow to occur, the wall shear stress must exceed the Bingham Plastic yield stress. Therefore, we must define some radius (R_y) where the shear stress will exceed the yield stress so flow can occur.

$$\tau_y = \frac{R_y}{2}\frac{dP}{dz}$$

In terms of our velocity profile, the fluid velocity will be constant from R=0 to R=R_y and written as

$$u_y = \frac{1}{4\mu}\frac{dP}{dz}(R^2 - R_y^{\,2}) - \frac{\tau_y}{\mu}(R - R_y)$$

Substitute for τ_y,

$$u_y = \frac{1}{4\mu}\frac{dP}{dz}(R^2 - R_y^{\,2}) - \frac{R_y}{2\mu}\frac{dP}{dz}(R - R_y)$$

So, the velocity profile in the outer region (from R_y to R) is

$$u = \frac{1}{\mu}\frac{dP}{dz}\left[\frac{(R^2 - r^2)}{4} - \frac{R_y}{2}(R - r)\right]$$

So, for the flow rate, this must be looked as two separate regions as well where there is the central region of flow and aouter region of flow.

$$Q_{total} = Q_{central} + Q_{peripheral}$$

Let's take these two separately.

$Q_{central}$ is easy since we know the velocity is constant through this region. Thus, the equation becomes

$Q_{central} = u_y \pi R_y^{\,2}$

Therefore,

$$Q_{central} = \frac{\pi}{\mu}\frac{dP}{dz}\left[\frac{(R^2 - R_y^{\,2})R_y^{\,2}}{4} - \frac{R_y^{\,3}}{2}(R - R_y)\right]$$

In the outer region, the flow rate is defined by an integral over that region,

$$Q_{peripheral} = \int_{R_y}^{R} 2\pi r u\,dr$$

After we substitute for u, the equation becomes

$$Q_{peripheral} = \int_{R_y}^{R} 2\pi r \frac{1}{\mu}\frac{dP}{dz}\left[\frac{(R^2 - r^2)}{4} - \frac{R_y}{2}(R - r)\right]dr$$

Integrating yields,

$$Q_{peripheral} = \frac{\pi}{\mu}\frac{dP}{dz}\left[\frac{R^4}{8} - \frac{R^2 R_y^{\,2}}{4} - \frac{5}{24}R_y^{\,4} - \frac{R^3 R_y}{6} + \frac{R R_y^{\,3}}{2}\right]$$

Combining both flow rate equations,

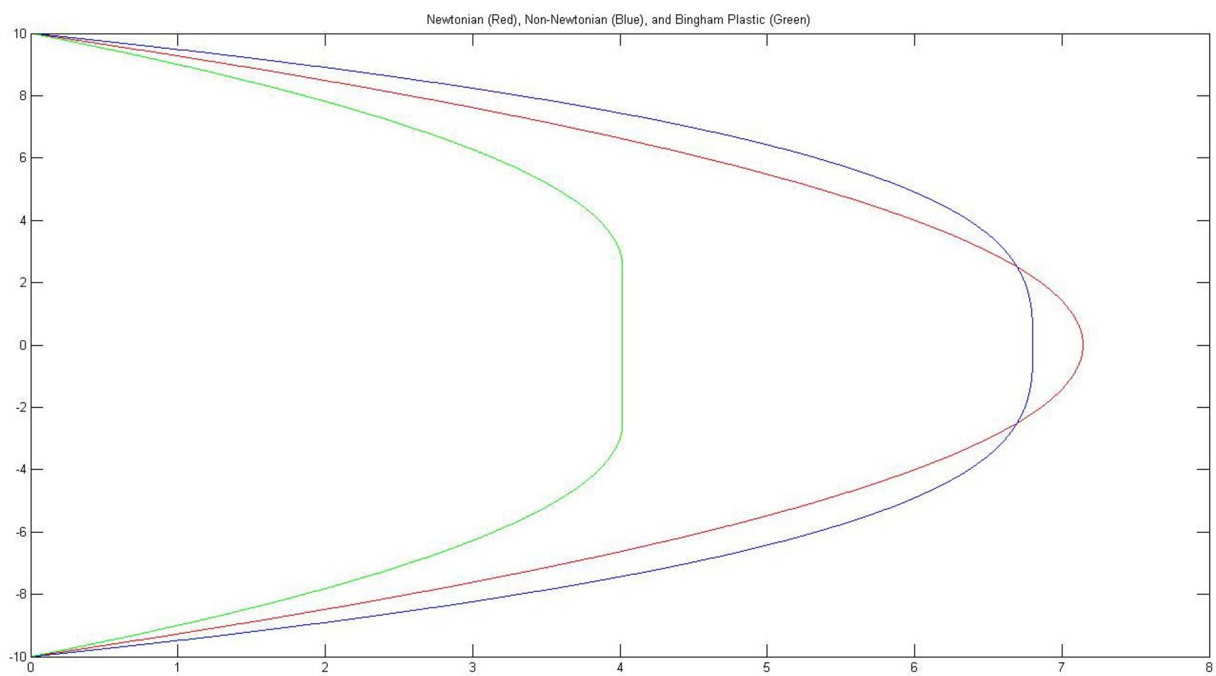
$$Q_{total} = \frac{\pi}{\mu}\frac{dP}{dz}\left[\frac{R^4}{8} + \frac{R_y^{\,4}}{24} - \frac{R^3 R_y}{6}\right]$$

If we then substitute back in for R_y, the following equation is developed,

$$Q_{total} = \frac{\pi R^4 dP}{8\mu dz}\left[1 - \frac{4}{3}\left(\frac{2\tau_y}{R\left(dP/_{dz}\right)}\right) + \frac{1}{3}\left(\frac{2\tau_y}{R\left(dP/_{dz}\right)}\right)\right]$$

Note: If the yield stress does not exist ($\tau_y = 0$), the equation reduces to that for Poiseuille Flow for a Newtonian Fluid.

The velocity profiles then become for Pseudoplastic, Newtonian, and Bingham Plastic fluids,



Fundamentals of Engineering (FE) Exam Problems

9-128

The continuity equation is also known as

(*a*) Conservation of mass (*b*) Conservation of energy (*c*) Conservation of momentum
(*d*) Newton's second law (*e*) Cauchy's equation

Answer (*a*) Conservation of mass

9-129

The Navier-Stokes equation is also known as

(*a*) Newton's first law (*b*) Newton's second law (*c*) Newton's third law
(*d*) Continuity equation (*e*) Energy equation

Answer (*b*) Newton's second law

9-130

Which one is the general differential equation form of the continuity equation for a control volume?

(*a*) $\int_{\text{CS}} \rho \vec{V} \cdot \vec{n}\, dA = 0$ (*b*) $\int_{\text{CV}} \frac{\partial \rho}{\partial t} dV + \int_{\text{CS}} \rho \vec{V} \cdot \vec{n}\, dA = 0$ (*c*) $\vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$

(*d*) $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$ (*e*) None of these

Answer (*d*) $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$

9-131

Which one is differential, incompressible, two-dimensional continuity equation in Cartesian coordinates?

(*a*) $\int_{CS} \rho \vec{V} \cdot \vec{n}\, dA = 0$ (*b*) $\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta} = 0$ (*c*) $\vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$

(*d*) $\vec{\nabla} \cdot \vec{V} = 0$ (*e*) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Answer (*e*) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

9-132

A steady velocity field is given by $\vec{V} = (u, v, w) = 2ax^2 y\vec{i} + 3bxy^2\vec{j} + cy\vec{k}$, where *a*, *b*, and *c* are constants. Under what conditions is this flow field incompressible?

(*a*) $a = b$ (*b*) $a = -b$ (*c*) $2a = -3b$ (*d*) $3a = 2b$ (*e*) $a = 2b$

Answer (*c*) $2a = -3b$

Solution

du/dx + dv/dy = 0 → 4axy + 6bxy = 0 → 4a = -6b → 2a = -3b

9-133

A steady, two-dimensional, incompressible flow field in the *xy*-plane has a stream function given by $\psi = ax^2 + by^2 + cy$, where *a*, *b*, and *c* are constants. The expression for the velocity component *u* is

(*a*) $2ax$ (*b*) $2by + c$ (*c*) $-2ax$ (*d*) $-2by-c$ (*e*) $2ax+ 2by + c$

Answer (*b*) $2by + c$

Solution

u = d(psi)/dy = 2by + c

9-134

A steady, two-dimensional, incompressible flow field in the *xy*-plane has a stream function given by $\psi = ax^2 + by^2 + cy$, where *a*, *b*, and *c* are constants. The expression for the velocity component *v* is

(*a*) $2ax$ (*b*) $2by + c$ (*c*) $-2ax$ (*d*) $-2by-c$ (*e*) $2ax + 2by + c$

Answer (*c*) $-2ax$

Solution

v = - d(psi)/dx = -2ax

9-135

If a fluid flow is both incompressible and isothermal, which property is not expected to be constant?

(*a*) Temperature (*b*) Density (*c*) Dynamic viscosity

(*d*) Kinematic viscosity (*e*) Specific heat

Answer (*e*) Specific heat

9-136

Which one is incompressible Navier-Stokes equation with constant viscosity?

(*a*) $\rho \frac{D\vec{V}}{Dt} + \vec{\nabla} P - \rho \vec{g} = 0$ (*b*) $-\vec{\nabla} P + \rho \vec{g} + \mu \vec{\nabla}^2 \vec{V} = 0$ (*c*) $\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \mu \vec{\nabla}^2 \vec{V}$

(*d*) $\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \vec{\nabla}^2 \vec{V}$ (*e*) $\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \vec{\nabla}^2 \vec{V} + \vec{\nabla} \cdot \vec{V} = 0$

Answer (*d*) $\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \vec{\nabla}^2 \vec{V}$

9-137

Which one is not correct regarding the Navier-Stokes equation?

(*a*) Nonlinear equation (*b*) Unsteady equation (*c*) Second-order equation

(*d*) Partial differential equation (*e*) None of these

Answer (*e*) None of these

9-138

In fluid flow analyses, which boundary condition can be expressed as $\vec{V}_{\text{fluid}} = \vec{V}_{\text{wall}}$

(*a*) No-slip (*b*) Interface (*c*) Free-surface (*d*) Symmetry (*e*) Inlet

Answer (*a*) No-slip