3 ROBOT KINEMATICS

Purpose:

The purpose of this chapter is to introduce you to robot kinematics, and the concepts related to both open and closed kinematics chains. Forward kinematics is distinguished from inverse kinematics.

3.1 Kinematics Chains

Mechanisms can be configured as kinematics chains. The chain is closed when the ground link begins and ends the chain; otherwise, it is open.

3.1.1 Serial robots

The manipulator of a serial robot is, in general, an open kinematics chain. The joints must be controlled individually.

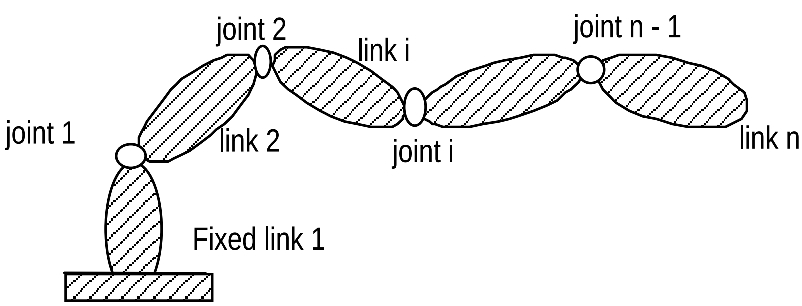


Figure 3-1 Set of serial links connected by joints

Assuming binary pair joints (joints supporting 2 links), the degrees-of-freedom (F) of a mechanism is governed by the equation

$$F = \lambda\,(n - 1) - \sum_{i=1}^{j} c_i \tag{3.1}$$

where

F = mechanism degrees-of-freedom

n = number of mechanism links

j = number of mechanism joints

c_i = number of constraints imposed by joint i

f_i = degrees-of-freedom permitted by joint i

j_i = number of joints with i degrees-of-freedom

λ = degrees-of-freedom in space in which mechanism functions

It is also true that

$$\lambda = c_i + f_i \quad (3.2)$$

which leads to *Grubler's Citerion:*

$$F = \lambda\,(n - j - 1) - \sum_{i=1}^{j} f_i \quad (3.3)$$

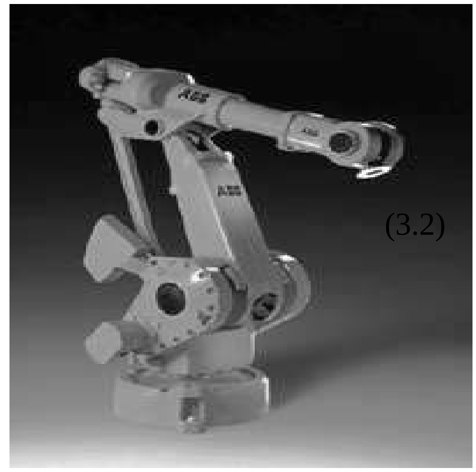


Figure 3-2 - ABB 6-axis robot

***Example** - 6-axis revolute robot(ABB IRB 4400):*

Using (3.1) and referencing Figure 3-2:

$F = 6\,(7 - 1) - 6\,(5) = 6$ *"as expected"*

Note: that the degrees-of-freedom of the robot equals the number of moving links, which equals the number of joints. To specify a unique manipulator configuration, each joint must be controlled.

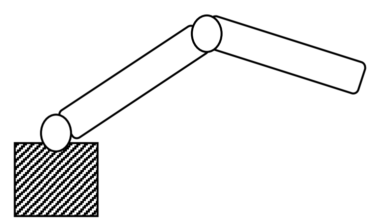


Figure 3-3 - Planar robot

***Example** - 3 axis revolute planar robot:*

Using (3.1) and referencing Figure 3-3:

$F = 3\,(3 - 1) - 2\,(2) = 2$

Why isn't the answer 3?

3.1.2 Redundant degrees-of-freedom

Grubler's Criterion is valid as long as there are no redundant joints. A redundant joint is one that is unnecessary because other joints can provide the needed position and/or orientation (see last 3 joints on IRB 4400).

Redundant joints can generate passive degrees-of-freedom, which must be subtracted from Grubler's equation to get

$$F = \lambda\,(n - j - 1) - \sum_{i=1}^{j} f_i - f_p \qquad (3.4)$$

Example: Quicktime video of robot. Shows how last joints can be configured to avoid redundancy during robot task motion.

3.1.3 Loop Mobility Criterion

Consider Figure 3-4. Some of the links have more than two joints, leading to multiple loops. The number of independent loops is the total number of loops excluding the external loop. For multiple loop chains it is true that j = n + L -1 which gives Euler's equation:

$$L = j - n + 1 \qquad (3.5)$$

Figure 3-4 applies this equation for a 2 loop mechanism.

Combining (3.5) with Grubler's Criterion, we get the *Loop Mobility Criterion:*

$$\sum f_i = F + \lambda L \qquad (3.6)$$

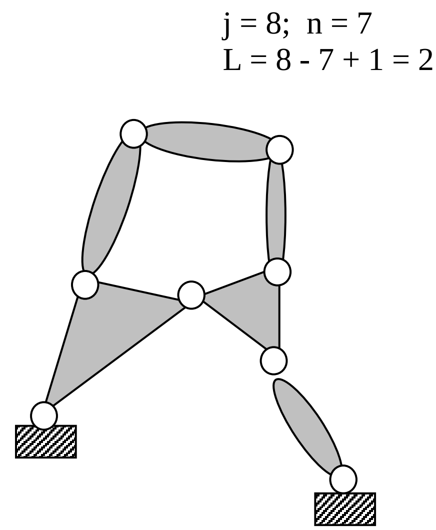


Figure 3-4 - Multiple loops

3.1.4 Parallel robots

A parallel robot is a closed loop chain, whereas a serial robot is an open loop chain. A hybrid mechanism is one with both closed and open chains.

Example - Figure 3-5 shows the Stewart-Gough platform. Determine the degrees-of-freedom. Note that each S-P-S combination generates a passive degree-of-freedom. Thus,

$\lambda = 6$; $n = 14$; $j_1 = 6$; $j_3 = 12$;

$f_p = 6$

Then,

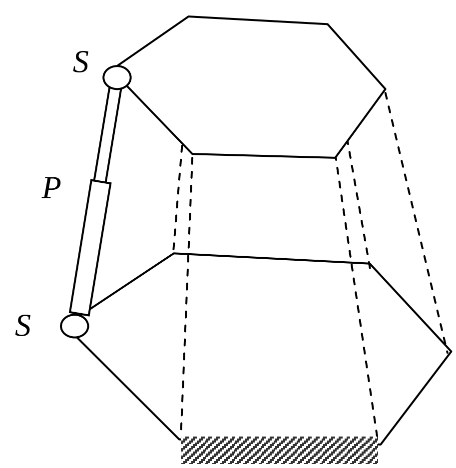


Figure 3-5 - Stewart-Gough Platform (note that dashed lines represent same S-P-S joint combination as shown: S = spherical joint; P = prismatic joint)

$F = 6(14 - 18 - 1) + (12x3 + 6) - 6 = \mathbf{6!}$ *As expected!*

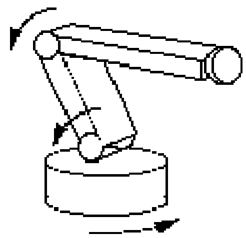


Figure 3-6 Revolute

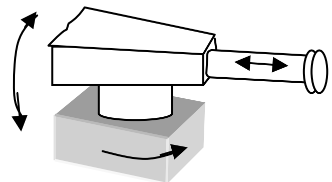


Figure 3-7 Spherical

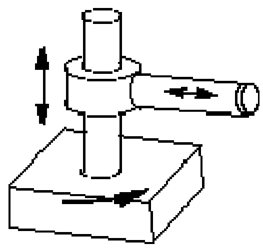


Figure 3-8 Cylindrical

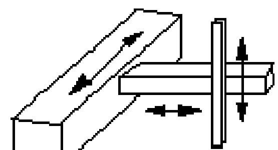


Figure 3-9 Rectangular

3.2 Serial Robot Types

Serial robots can be classified as revolute, spherical, cylindrical, or rectangular (translational, prismatic, or Cartesian). These classifications describe the primary DOF (degrees-of-freedom) which accomplish the global motion as opposed to the distal (final) joints that accomplish the local, primarily orientation, motion.

3.3 Serial Robot Types

There are numerous parallel robot types. Some of these will be examined later.

3.4 Open Chain Link Coordinates

According to the conventional Denavit-Hartenberg (D-H) notation (Denavit, J. and Hartenberg, "A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices," J. of Applied Mechanics, June, 1955, pp. 215-221.), only four parameters (a, d, θ, α) are necessary to define a frame in space (or joint axis) relative to a reference frame:

a = minimum distance between line L (the z axis of next frame) and z axis (mutually orthogonal line between line L and z axis)

d = distance along z axis from z origin to minimum distance intersection point

θ = angle between x-z plane and plane containing z axis and minimum distance line

α = angle between z axis and L

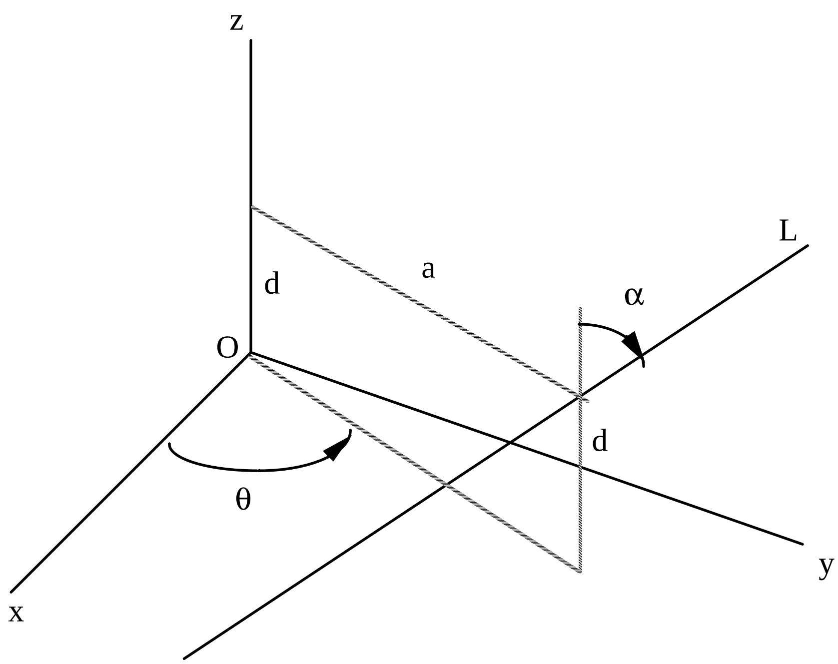


Figure 3-10 Conventional D-H parameters

Alternatively we can define a line by any point P on the line and its direction unit vector **n**. This requires 5 parameters since $n_x^2 + n_y^2 + n_z^2 = 1$.

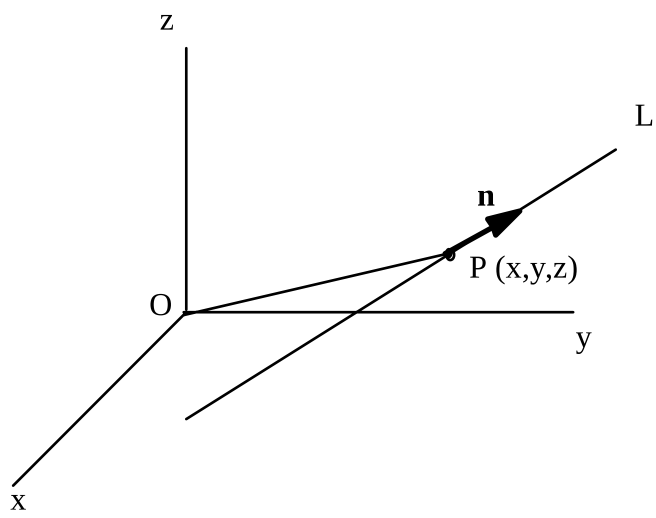


Figure 3-11 Point vector line description

The Denavit-Hartenberg parametric description of lines can be extended to represent frame coordinates for a kinematic chain of revolute and translational joints - consider the figure below. Note that there are several forms of these parameters being applied to the forward and inverse kinematics of serial mechanisms.

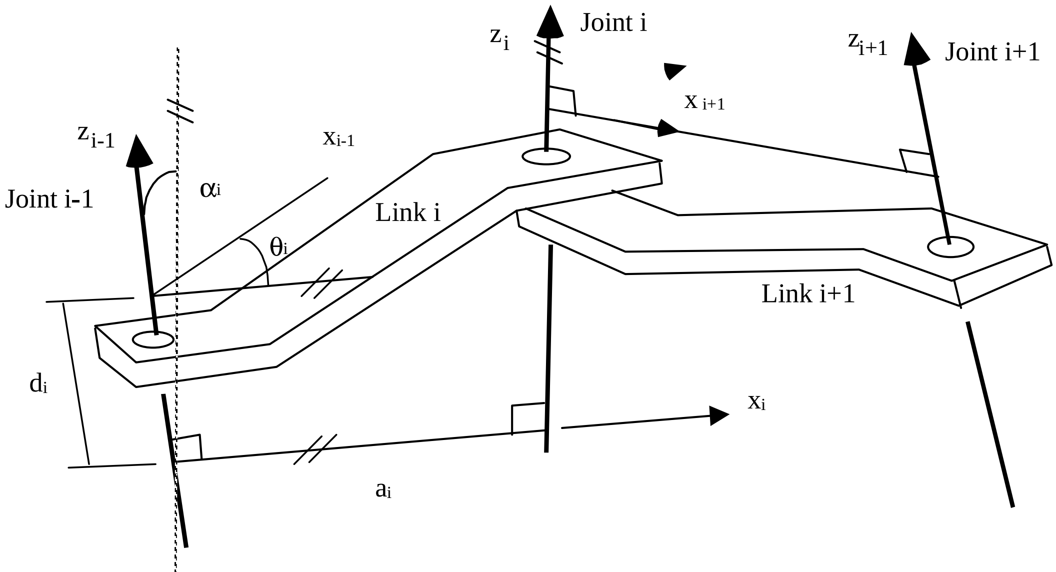


Figure 3-12 Conventional D-H notation for serial links/joints

Each link i has an inward joint i and an outward joint i +1. The coordinate system is established beginning at joint 1, the input joint, and numbering outward. For a revolute robot the coordinate z axis for each link lies colinear with the axis of rotation. The x axis is established by the miminum distance line between the current z axis and the z axis of the inner joint. a is the minimum distance between the two consecutive z axes. The axes are numbered such that the i -1 axes are associated with the i^{th} joint of the i^{th} link (and thus describes the displacement of the previous link). The set of axes established for a PUMA robot is shown as follows.

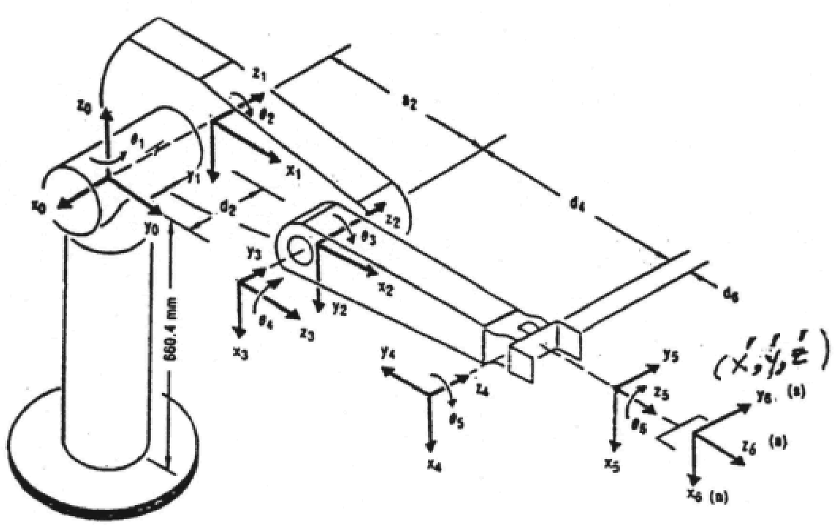


Figure 3- 13 Puma robot

Joint	a_i	d_i	θ_i	α_i	Range
1	0	0	90	-90	-150 to 150
2	432	149.5	0	0	-225 to 45
3	0	0	90	90	-45 to 225
4	0	432	0	-90	-110 to 170
5	0	0	0	90	-100 to 100
6	0	55.5	0	0	-265 to 265

Figure 3-14 PUMA D-H parameters

Using the D-H representation, the four parameters are described as

θ_i = joint angle of x_i axis relative to x_{i-1} axis with defined according to RH rule about z_{i-1} axis.

d_i = distance from the origin of the i-1 axes to the intersection of the z_{i-1} axis with the x_i axis and measured along the z_{i-1} axis.

a_i = minimum distance between z_{i-1} and z_i.

α_i = offset angle of z_i axis relative to z_{i-1} axis measured about the x_i axis using RH rule.

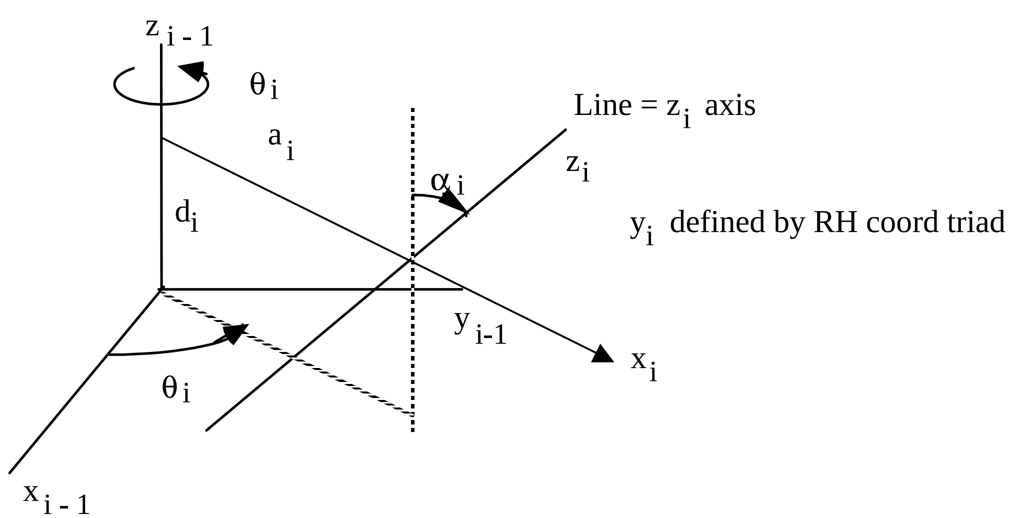


Figure 3-15 D-H frame notation

For revolute joints θ_i is the joint variable with d_i, a_i, and α_i constant. For prismatic joints the joint variable is d_i with θ_i, a_i, and α_i constant (a_i is typically zero)

Given a revolute joint a point $\mathbf{x}_i$ located on the i^{th} link can be located in i - 1 axes by the following transformation set which consist of four homogeneous transformations (2 rotations and 2 translations). The set that will accomplish this is

$$\mathbf{A}_i = \mathbf{H}(d,z_{i-1})\,\mathbf{H}(\theta,z_{i-1})\,\mathbf{H}(a,x_i)\,\mathbf{H}(\alpha,x_i) \qquad (i = 1, ...n) \tag{3.7}$$

where

$$\mathbf{H}(\alpha, x_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}(a, x_i) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}(\theta, z_{i-1}) = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}(d, z_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying the matrix multiplication of (3.2),

$$\mathbf{A}_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -s\alpha_i c\theta_i & a_i c\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.8}$$

Class problem: What is the correct order in multiplying the four **H** transformation matrices to get (3.8)?

$\mathbf{A}_i =$??

3.4.1 Other D-H Notation

The CODE system uses a set of D-H parameters that differ from the conventional set. These are similar to Craig's D-H convention.

Referring to Figure 3.16, we note that four parameters must be specified:

a_i = minimum distance between joint i axis (z_i) and joint i-1 axis (z_{i-1})

d_i = distance from minimum distance line (x_{i-1} axis) to origin of ith joint frame measured along z_i axis.

α_i = angle between z_i and z_{i-1} measured about previous joint frame x_{i-1} axis.

θ_i = angle about z_i joint axis which rotates x_{i-1} to x_i axis in right hand sense.

The x_i axis is the minimum distance line defined from z_i to z_{i+1}; z_i is defined as the joint rotation or translation axis axis and y_i by the right hand rule ($\mathbf{z}_i$ x $\mathbf{x}_i$). The origin of each joint frame is defined by the minimum distance line intersection on the joint axis.

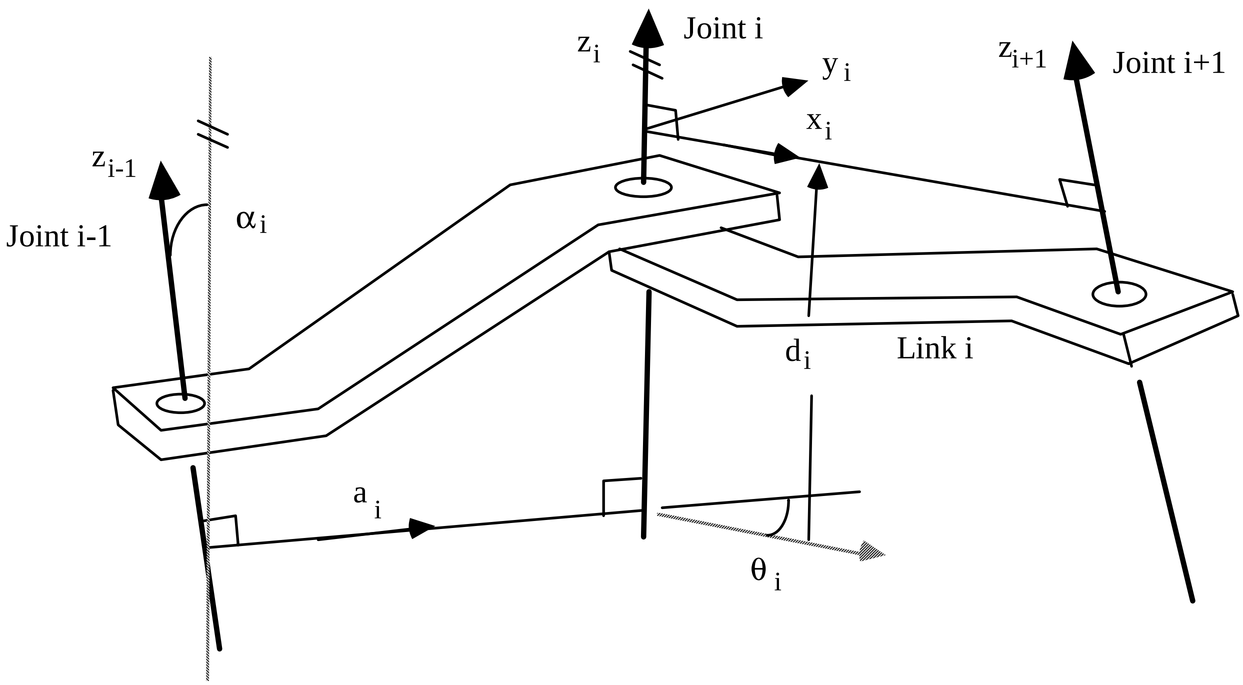


Figure 3-16 Revised D-H parameters

The transformation for this set of D-H parameters is

$$\mathbf{A}_i = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_i \\ s\theta_i c\alpha_i & c\theta_i c\alpha_i & -s\alpha_i & -s\alpha_i d_i \\ s\theta_i s\alpha_i & c\theta_i s\alpha_i & c\alpha_i & c\alpha_i d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

Class problem: derive the set of D-H parameters for the Puma robot being considered.

3.5 Forward Kinematics for Serial Robots

Given the **A** transformation matrices of one joint axes relative to the preceding axes, one can relate any point in the i^{th} link to the global reference frame by the following transformation set. Let $\mathbf{v}_i$ be a point fixed to the i^{th} link. Its coordinates $\mathbf{u}_i$ in global axes are (n = # DOF)

$$\mathbf{u}_i = \mathbf{A}_1\, \mathbf{A}_2 \ldots \mathbf{A}_i\, \mathbf{v}_i \qquad (i = 1, 2, \ldots n) \tag{3.10}$$

Typically we represent the set of transformations above by a single matrix called the **T** matrix

$$\mathbf{T}i = \mathbf{A}_1\, \mathbf{A}_2 \ldots \mathbf{A}_i = \prod_{j=11}^{i} \mathbf{A}_{\mathbf{j}} \tag{3.11}$$

The **T** matrix locating the gripper frame is

$$\mathbf{T}_n = \prod_{j=11}^{n} \mathbf{A}_{\mathbf{j}} \tag{3.12}$$

The subscript n may be dropped for simplicity.

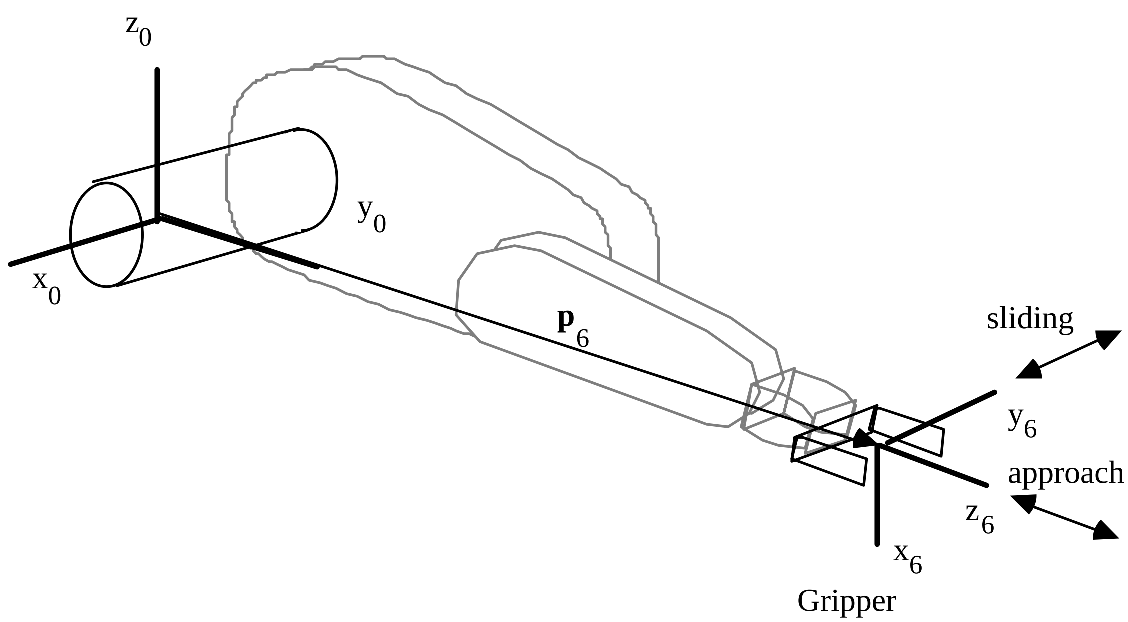


Figure 3-17 The gripper frame

Examining the gripper coordinates in the PUMA figure shown previously (x_6, y_6, z_6), the z_6 axis will typically denote the gripper approach direction while the y_6 axis denotes the sliding direction. Examining **T**, the first 3 columns describe the frame direction cosines of x_6, y_6, z_6 relative to global (or base) frame whereas the 4th column locates the x_6, y_6, z_6 origin relative to global frame.

3.5.1 Forward kinematics using alternative D-H notation

Using homogeneous transformations between the serial links of a robot the pose of a tool frame at the end of the robot can be determined by the equation

$$\mathbf{T} = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 ... \mathbf{A}_n\mathbf{G} \tag{3.13}$$

where **T** locates the tool relative to the robot base frame and **G** locates the tool relative to the last joint/link frame. Note that the joint frames are usually oriented such that the rotation or translation takes place about the joint z axis.

***Question**: Why is **G** required in the alternative notation, but not in the original D-H notation?*

In forward kinematics the joint translation or rotation is specified directly and the tool is commanded to the pose described mathematically by **T** since each $\mathbf{A}_i$ is known. Forward kinematics is used in teach pendant programming.

3.6 Inverse Kinematics for Serial Robots

Inverse kinematics raises the opposite question: *Given that I know the desired pose of the tool, what are the joint values required to move the tool to the pose?*

Mathematically, we rearrange equation (3.13) so that we isolate the homogeneous transformations that are a function of the unknown joint values and somehow solve for the joint values by applying the following equation:

$$\text{(joint values unknown)} \quad \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3...\mathbf{A}_n = \mathbf{G}^{-1}\mathbf{T} \quad \text{(right side known)} \tag{3.14}$$

Inverse kinematics is used for controlling path following or in sensor directed motion where a target can be determined.

3.6.1 Inverse Kinematics – An Example

Since a function of the robot is to place objects in positions and orientations described in global space, it is desired to determine the joint variables to accomplish this. This is known as the inverse kinematics (invkin) problem. This section considers the inverse kinematics solution for the PUMA manipulator.

The open loop equation for a six axis robot like the Puma is $\mathbf{T} = \mathbf{A}_1\ \mathbf{A}_2....\mathbf{A}_6$. where the target pose $\mathbf{T}$ is known and the joint variables (θ_i in this case) which make up $\mathbf{A}_i$ matrices are unknown (and to be found).

The solution, calculated in two stages, first uses a position vector from the waist to the wrist. This vector allows for the solution of the first three primary DOF that accomplish the global motion. The last 3 DOF (secondary DOF) are found using the calculated values of the first 3 DOF and the orientation matrices $\mathbf{T}_4$, $\mathbf{T}_5$, and $\mathbf{T}_6$.

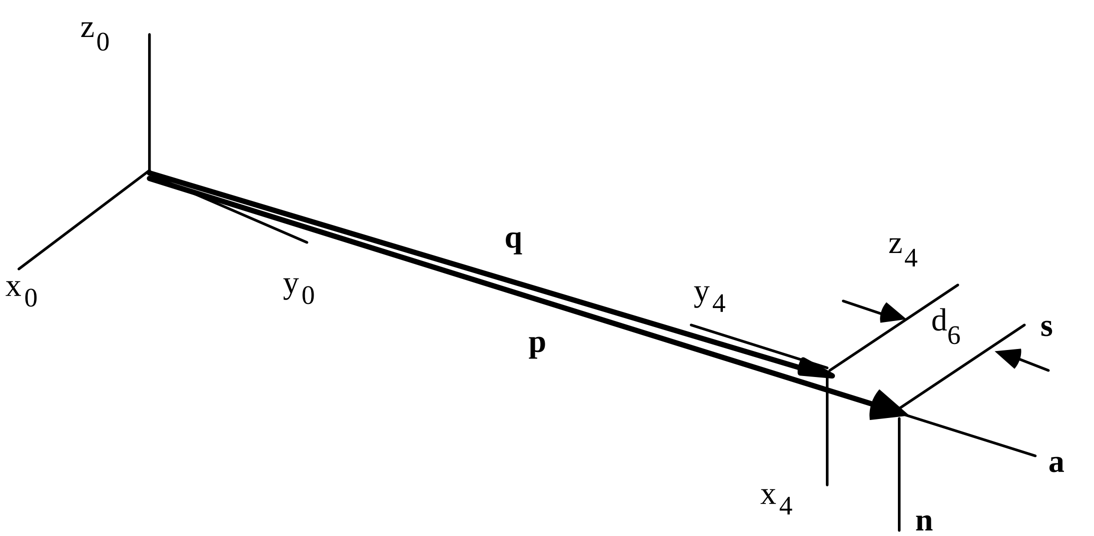


Figure 3-18 Waist to wrist solution

Let the gripper frame be defined by the unit vector triad **n**, **a**, and **s** (solution procedure in Paul's textbook *Robot Manipulators*)

$$\mathbf{q} = \mathbf{p} - d_6\mathbf{a} \tag{3.15}$$

It is assumed that **p** known and **a** known, since for a PUMA manipulator, **T** (the desired global frame of the gripper) is of the form:

$$\mathbf{T} = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.16}$$

where (e.g., $C_{23} = \cos(\theta_2 + \theta_3)$)

$$n_x = C_1[C_{23}(C_4\ C_5\ C_6 - S_4\ S_6) - S_{23}\ S_5\ C_6] - S_1[S_4\ C_5\ C_6 + C_4\ S_6] \tag{3.17a}$$

$$n_y = S_1[C_{23}(C_4\ C_5\ C_6 - S_4\ S_6) - S_{23}\ S_5\ C_6] + C_1[S_4\ C_5\ C_6 + C_4\ S_6] \tag{3.17b}$$

$$n_z = -S_{23}[C_4\ C_5\ C_6 - S_4\ S_6] - C_{23}\ S_5\ C_6 \tag{3.17c}$$

$$s_x = C_1[-C_{23}(C_4\ C_5\ S_6 + S_4\ C_6) + S_{23}\ S_5\ S_6] - S_1[-S_4\ C_5\ S_6 + C_4\ C_6] \tag{3.17d}$$

$$s_y = S_1[-C_{23}(C_4\ C_5\ S_6 + S_4\ C_6) + S_{23}\ S_5\ S_6] + C_1[-S_4\ C_5\ S_6 + C_4\ C_6] \tag{3.17e}$$

$$s_z = S_{23}(C_4\ C_5\ S_6 + S_4\ C_6) + C_{23}\ S_5\ S_6 \tag{3.17f}$$

$$a_x = C_1(C_{23}\ C_4\ S_5 + S_{23}\ C_5) - S_1\ S_4\ S_5 \tag{3.17g}$$

$$a_y = S_1(C_{23}\ C_4\ S_5 + S_{23}\ C_5) + C_1\ S_4\ S_5 \tag{3.17h}$$

$$a_z = -S_{23}\ C_4\ S_5 + C_{23}\ C_5 \tag{3.17i}$$

$$p_x = C_1\,[d_6(C_{23}\ C_4\ S_5 + S_{23}\ C_5) + S_{23}\ d_4 + a_2\ C_2] - S_1(d_6\ S_4\ S_5 + d_2) \tag{3.17j}$$

$$p_y = S_1\,[d_6(C_{23}\ C_4\ S_5 + S_{23}\ C_5) + S_{23}\ d_4 + a_2\ C_2] + C_1(d_6\ S_4\ S_5 + d_2) \tag{3.17k}$$

$$p_z = d_6\,(C_{23}\ C_5 - S_{23}\ C_4\ S_5) + C_{23}\ d_4 - a_2\ S_2 \tag{3.17l}$$

Now setting $\theta_4 = \theta_5 = \theta_6 = 0$ and $d_6 = 0$ we get the components of **q** from (3.17j) - (3.17l).

$$\mathbf{q} = \mathbf{p}\,\Big]_{\theta_4 = \theta_5 = \theta_6 = d_6 = 0}$$

or by applying (3.8)

$$q_x = C_1 (d_4 S_{23} + a_2 C_2) - d_2 S_1 \quad (3.18a)$$
$$q_y = S_1 (d_4 S_{23} + a_2 C_2) + d_2 C_1 \quad (3.18b)$$
$$q_z = d_4 C_{23} - a_2 S_2 \quad (3.18c)$$

Now θ, can be determined from q_x and q_y components. Let $\lambda = d_4 S_{23} + a_2 C_2$; thus,

$$q_x = C_1 \lambda - d_2 S_1$$

$$q_y = S_1 \lambda + d_2 C_1$$

It can be shown that $\lambda = \pm\sqrt{q_x^2 + q_y^2 - d_2^2}$ and that

$$\theta_1 = \tan^{-1}\left[\frac{\lambda q_y - d_2 q_x}{\lambda q_x + d_2 q_y}\right] \qquad \text{(use atan2 } (-\pi \le \theta \le \pi)) \qquad (3.19)$$

where (3.19) is calculated using the four quadrant atan2 function. One notes from (3.19) that 2 solutions exist: + for <u>left shoulder</u> PUMA; - for <u>right shoulder</u> PUMA. The solution for θ_3 can be found by squaring the (3.18) components and adding to find sin θ_3, then finding

$$\cos\theta_3 = \text{sqrt}[1 - \sin^2\theta_3]$$
$$\theta_3 = \tan^{-1}\left[\frac{q_x^2 + q_y^2 + q_z^2 - d_4^2 - a_2^2 - d_2^2}{\pm\sqrt{4d_4^2 a_2^2 - (q_z^2 + q_y^2 + q_z^2 - d_4^2 - a_2^2 - d_2^2)^2}}\right] \qquad (3.20)$$

The + soln is for the elbow above hand whereas the - soln is for the elbow below hand.

Now, given θ_3 (and S_3 and C_3), we can expand C_{23} and S_{23} and finally arrive at (use atan2)

$$\theta_2 = \tan^{-1}\left[\frac{-\left[q_z (a_2 + d_4 S_3) - d_4 C_3 \left(\pm\sqrt{q_x^2 + q_y^2 - d_2^2}\right)\right]}{q_z d_4 C_3 - (a_2 + d_4 S_3)\left(\pm\sqrt{q_x^2 + q_y^2 - d_2^2}\right)}\right] \qquad (3.21)$$

The - soln corresponds to the left arm configuration, + soln corresponds to right arm configuration.

Obviously knowing θ_1, θ_2, and θ_3 permits definition of ${}^0\mathbf{T}_3$. To determine θ_4, θ_5, and θ_6 we assume that an approach direction is known (**a** known) and that hand orientation is specified (**n**, **s**). For the PUMA robot we can arrange the joint axes such that

$$\mathbf{z}_4 = \frac{\pm(\mathbf{z}_3 \times \mathbf{a})}{\|\mathbf{z}_3 \times \mathbf{a}\|} \qquad (\mathbf{z}_4 \text{ axis direction cosines}) \tag{3.22}$$

$$\mathbf{z}_5 = \mathbf{a} \qquad (\mathbf{z}_5 \text{ axis direction cosines}) \tag{3.23}$$

$$\mathbf{y}_6 = \mathbf{s} \qquad (\mathbf{y}_6 \text{ axis direction cosines}) \tag{3.24}$$

Now given the above criteria, we can solve for θ_4 from

$$C_4 = \mathbf{y}_3 \cdot \mathbf{z}_4 \qquad (= \mathbf{y}_3^{\,T} \mathbf{z}_4)$$
$$S_4 = -\mathbf{x}_3 \cdot \mathbf{z}_4 \qquad (= -\mathbf{x}_3^{\,T} \mathbf{z}_4)$$

Determine $\mathbf{x}_3$ and $\mathbf{y}_3$ from 1st and 2nd columns of ${}^0\mathbf{T}_3$ to get (use atan2)

$$\theta_4 = \tan^{-1}\left(\frac{c_1\, a_y - s_1\, a_x}{c_1\, c_{23}\, a_x + s_1\, c_{23}\, a_y - s_{23}\, a_z}\right) \tag{3.25}$$

Given θ_4 we can determine ${}^0\mathbf{T}_4$. In a similar fashion as for θ_4 we can determine θ_5.

Now given θ_4, ${}^0\mathbf{T}_4$ is defined (and so is $\mathbf{x}_4$, $\mathbf{y}_4$, and $\mathbf{z}_4$).

Now $S_5 = \mathbf{x}_4 \cdot \mathbf{a}$ and $C_5 = -\mathbf{y}_4 \cdot \mathbf{a}$ so that (use atan2)

$$\theta_5 = \tan^{-1}\left[\frac{(c_1\, c_{23}\, c_4 - s_1\, s_4)\, a_x + (s_1\, c_{23}\, c_4 + c_1\, s_4)\, a_y - c_4\, s_{23}\, a_z}{c_1\, s_{23}\, a_x + s_1\, s_{23}\, a_y + c_{23}\, a_z}\right] \tag{3.26}$$

Now if $\theta_5 \approx 0$, a degenerate case results in which a 5-axis robot would be sufficient since joint 5 is not needed. To solve for θ_6 align $\mathbf{y}_6$ with **s** so that $S_6 = \mathbf{y}_5 \cdot \mathbf{n}$ and $C_6 = \mathbf{y}_5 \cdot \mathbf{s}$ where $\mathbf{y}_5$ comes from $\mathbf{T}_0^5$ and **n** and **s** come from **T**. We get (use atan2)

$$\theta_6 = \tan^{-1}\left[\frac{-(S_1C_4 + C_1C_{23}S_4)n_x + (-C_1C_4 - S_1C_{23}S_4)n_y + (S_4S_{23})n_z}{-(S_1C_4 + C_1C_{23}S_4)s_x + (C_1C_4 - S_1C_{23}S_4)s_y + (S_4S_{23})s_z}\right] \tag{3.27}$$

3.7 Kinematics Summary for Serial Robots

Both forward and inverse kinematics are important to robotics. The robot teach-pendant uses direct joint control to place the robot tool at desired poses in space. It is a form of forward kinematics control.

When the target for an end-effector tool is specified directly, either by a sensor or as the robot interpolates moves along specified curvilinear paths in space, it requires invkin solutions to generate the necessary joint values.

D-H parameters provide a simple way of relating joint frames relative to each other, although more than one D-H form proliferate the application methods. The invkin solutions can be complex depending on the robot structure.

3.8 Forward Kinematics for Parallel Robots

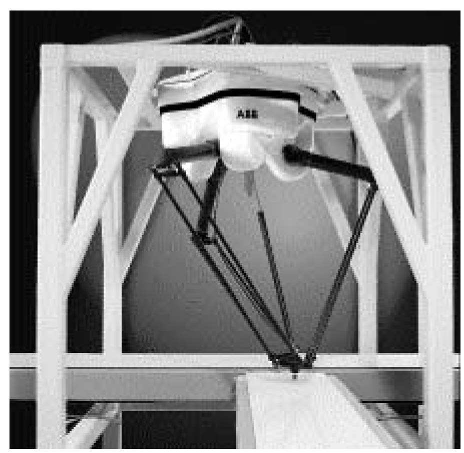
A parallel mechanism is symmetrical if:

- number of limbs is equal to the number of degrees-of-freedom of the moving platform
- joint type and joint sequence in each limb is the same
- number and location of the actuated joints is the same.

Otherwise, the mechanism is asymmetrical. We will examine the kinematics for symmetrical mechanisms.

We define several terms:

limb = a serial combination of links and joints between ground and the moving rigid platform
connectivity of a limb (C_k) = degrees-of-freedom associated with all joints in a limb



Observation of symmetrical mechanisms will establish that

$$\sum_{k=1}^{m} C_k = \sum_{i=1}^{j} f_i \qquad (3.28)$$

where j is the number of joints in the parallel mechanism and m is the number of limbs. It is also observed that the connectivity of each limb should not exceed the motion parameter (λ) and not be less than the degrees-of-freedom of the moving platform (F), leading to

$$\lambda \geq C_k \geq F \qquad (3.29)$$

Example - University of Maryland Mechanism (or ABB Picker robot)

The general degrees-of-freedom equation (3.3) does not apply to this robot because of the symmetry of the design and other constraints. This robot has 3 translational degrees-of-freedom, with a rotational orientation joint added to the center of the moving platform in the commercial version.

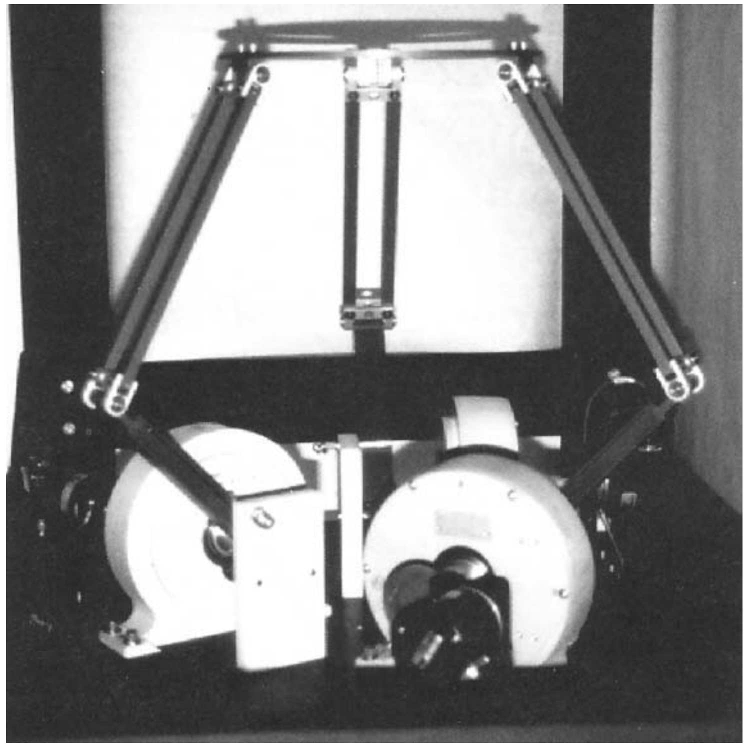


Figure 3-20 Maryland robot schematic

Figures 3-20 and 3-21 depict the notation for analysis of the robot kinematics. We define a limb coordinate system as x_i, y_i, z_i, orienting the limb base point A_i relative to the fixed robot base frame x,y,z by the orientation angle ϕ_i. The limb's revolute joints are labeled as θ_{1i}, θ_{2i}, etc.. where i defines the limb number.

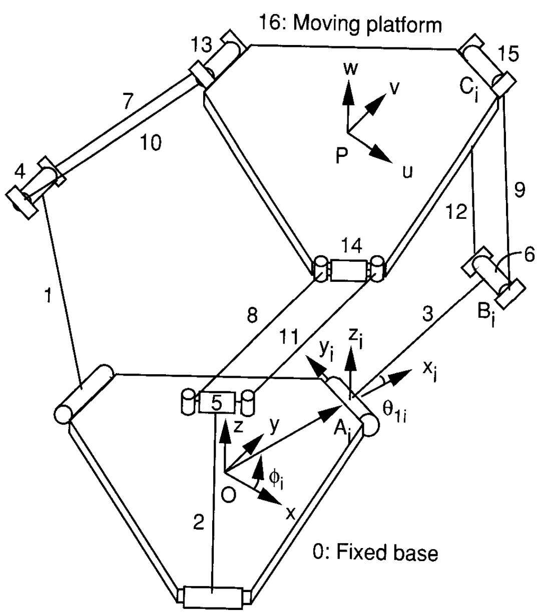


Figure 3-21 Maryland limb schematic

Examining Figures 3-20 and 3-21, we can write a loop closure equation:

$$\mathbf{AB}_i + \mathbf{BC}_i = \mathbf{OP} + \mathbf{PC}_i - \mathbf{OA}_i \qquad (3.30)$$

Expressing (3.30) in the limb i coordinate frame, we get

$$\begin{bmatrix} a\,c\theta_{1i} + b\,s\theta_{3i}\,c(\theta_{1i} + \theta_{2i}) \\ b\,c\theta_{3i} \\ a\,s\theta_{1i} + b\,s\theta_{3i}\,s(\theta_{1i} + \theta_{2i}) \end{bmatrix} = \begin{bmatrix} c_{xi} \\ c_{yi} \\ c_{zi} \end{bmatrix} \qquad (3.31)$$

$$\mathbf{c}_i = \begin{bmatrix} c_{xi} \\ c_{yi} \\ c_{zi} \end{bmatrix} = \begin{bmatrix} c\phi_i & s\phi_i & 0 \\ -s\phi_i & c\phi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \begin{bmatrix} h - r \\ 0 \\ 0 \end{bmatrix} \tag{3.32}$$

where $\mathbf{c}_i$ locates C_i relative to limb coordinate frame and $\mathbf{p}$ locates P relative to the x,y,z base frame. Note that θ_{3i} represents the out of plane motion of point C_i.

The forward kinematics (or direct kinematics as referred to in Tsai's book) can be determined by specifying the angles θ_{11}, θ_{12}, and θ_{13}. The problem is to determine the position x,y,z of point P.

First note that for a given angle θ_{1i} that point P will lie on a sphere centered at B_i', which is offset in the horizontal direction from C_i to P by a distance h, Figure 3-22.

Considering the 3 limbs P must lie on the intersection of three such spheres. There are four possibilities:

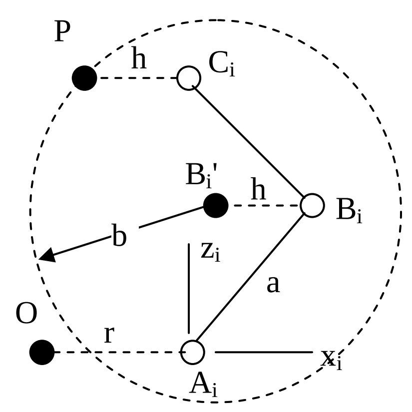


Figure 3-22 Forward solution

- *Generic solution* - spheres intersect at two points, giving two solutions (one sphere intersects circle of intersection of two other spheres)
- *Singular solution* - one sphere is tangent to circle of intersection of two other spheres
- *Singular solution* - center of any two of the three spheres coincide, resulting in an infinite number of solutions. The structural design will preclude this happening. An example of such a problem is when the independent joints are all $\pi/2$ and r = h.
- *No solution* - three spheres do not intersect

We can rearrange (3.31) and (3.32) to obtain

$$\begin{bmatrix} b\, s\theta_{3i}\, c(\theta_{1i} + \theta_{2i}) \\ b\, c\theta_{3i} \\ b\, s\theta_{3i}\, s(\theta_{1i} + \theta_{2i}) \end{bmatrix} = \begin{bmatrix} c\phi_i & s\phi_i & 0 \\ -s\phi_i & c\phi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \begin{bmatrix} -a\, c\theta_{1i} + h - r \\ 0 \\ -a\, s\theta_{1i} \end{bmatrix} \tag{3.33}$$

and then square the three components to get the equation:

$$b^2 = p_x^2 + p_y^2 + p_z^2 - 2(p_x\, c\phi_i + p_y\, s\phi_i)(a\, c\theta_{1i} + r - h) - 2p_z\, a\, s\theta_{1i} + (a\, c\theta_{1i} + r - h)^2 + a^2\, s^2\theta_{1i} \tag{3.34}$$

This equation represents the sphere for link i. We now apply (3.34) for links 1 and j (2 or 3) and then solve them simultaneously to get 2 equations (j = 2 and j =3):

$$e_{1j}\, p_x + e_{2j} + p_y + e_{3j}\, p_z + e_{4j} = 0 \qquad (j = 2,3) \tag{3.35}$$

where

$$e_{1j} = 2\, c\phi_j\, (a\, c\theta_{1j} + r - h) - 2\, c\phi_1\, (a\, c\theta_{11} + r - h) \tag{3.36}$$

$$e_{2j} = 2\, s\phi_j\, (a\, c\theta_{1j} + r - h) - 2\, s\phi_1\, (a\, c\theta_{11} + r - h) \tag{3.37}$$

$$e_{3j} = 2a\, s\theta_{1j} - 2a\, s\theta_{11} \tag{3.38}$$

$$e_{4j} = (a\, c\theta_{11} + r - h)^2 + a^2\, s^2\theta_{11} - (a\, c\theta_{1j} + r - h)^2 - a^2\, s^2\theta_{1j} \tag{3.39}$$

Note that the form in (3.36) is the equation of a plane since θ_{11} , θ_{12} ,and θ_{13} are known. If we generate (3.35) for both j = 2 and j = 3, then solve for the intersection of two planes, we get the equation of a line. We then intersect this line with one of the spheres to generate two solutions. Equivalently, the quadratic equation for these two solutions can be found by solving the two equations represented by (3.35) for p_y and p_z in terms of p_z and then substituting into (3.34) to get

$$k_0\, p_x^2 + k_1\, p_x + k_2 = 0 \tag{3.40}$$

where the quadratic coefficients are

$$k_o = 1 + l_1^2/\, l_2^2 + l_4^2/\, l_2^2$$

$$k_1 = 2\, l_0 l_1/l_2^2 + 2\, l_3 l_4/l_2^2 - 2 l_5\, c\phi_1 - 2\, l_5 l_1\, s\phi_1/l_2 - 2\, a l_4\, s\theta_{11}/l_2$$

$$k_2 = l_5^2 - b^2 + l_0^2/\, l_2^2 + l_3^2/\, l_2^2 + a^2\, s^2\theta_{11} - 2\, l_0 l_5\, s\phi_1/l_2 - 2\, a l_3\, s\theta_{11}/l_2$$

and

$$l_0 = e_{32}\, e_{43} - e_{33}\, e_{42}$$

$$l_1 = e_{13}\, e_{32} - e_{12}\, e_{33}$$

$$l_2 = e_{22}\, e_{33} - e_{23}\, e_{32}$$

$$l_3 = e_{23}\, e_{42} - e_{22}\, e_{43}$$

$$l_4 = e_{12}\, e_{23} - e_{13}\, e_{22}$$

$$l_5 = a\, c\theta_{11} + r - h$$

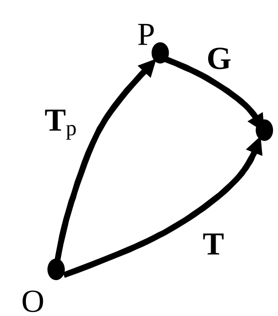


Figure 3-23 Closing the gripper to platform loop

The solution cases are

- $k_1^2 - 4k_0\ k_2 > 0$, two solutions
- $k_1^2 - 4k_0\ k_2 = 0$, one solution
- $k_1^2 - 4k_0\ k_2 < 0$, no solution

Once p_x is found in (3.40), then you determine p_y and p_z by back substitution into (3.35).

3.8.1 Forward Kinematics Implementation

How would you use a teach pendant to drive this robot? In reality you would probably not command the joints directly, but most likely command translations in the **u**, **v**, and **w** directions. **Thus, you would not likely drive this robot using forward kinematics but only apply inverse kinematics.**

3.9 Inverse Kinematics for the Maryland/Picker Parallel Robot

We assume that the position vector **p** is given. The problem is to find the joint angles to place point P at **p**. In reality, the gripper would not be located at P, but be attached to the moving platform. This is determined by gripper frame **G** relative to the platform coordinate axes.

A target frame is specified as **T**. We determine the target for the platform coordinate axes as shown in Figure 3-23. The frame for point P is determined from the fourth column of $\mathbf{T}_p = \mathbf{TG}^{-1}$. We designate this vector as **p**.

Given **p** we determine the location of point C_i. This is simple because the moving platform cannot rotate and thus the line between P and C_i translates only. Thus, given P (as determined by **p**) and the distance h, we can determine C_i as displaced from P by a vector of length h that is parallel to x_i.

The locus of motion of link B_iC_i is a sphere with center at C_i and radius b. The figure shown in the text as Figure 3.12 is deceiving because it is presented two-dimensionally. It can only be interpreted in 3-D.

From (3.31) we can determine two solutions for θ_{3i} as

$$\theta_{3i} = \cos^{-1}(c_{yi}/b) \qquad (3.41)$$

Tsai confuses the inverse kinematics solution, since you should only choose a positive solution for θ_{3i}.

Why?

Given θ_{3i} we can determine an equation for θ_{2i} by summing the squares of (3.31) to get

$$2ab\ s\theta_{3i}\ c\theta_{2i} + a^2 + b^2 = c_{xi}^2 + c_{yi}^2 + c_{zi}^2 \tag{3.42}$$

which leads to a solution for θ_{2i} as

$$\theta_{2i} = \cos^{-1}(\kappa) \tag{3.43}$$

where $\kappa = (c_{xi}^2 + c_{yi}^2 + c_{zi}^2 - a^2 - b^2)/(2ab\ s\theta_{3i})$. Physically, we can determine two solutions for θ_{2i} ("+" angle and "-" angle similar to elbow up/down case).

The two solutions for θ_{1i} can be determined from (3.31) by expanding the double angle formulas, solving for the sine and cosine of θ_{1i} and then using the atan2 function to get θ_{1i}.

It is possible that the target frame may fall outside the robot's reach; thus, we must examine the special cases:

- *Generic solution* - circle of link AB intersects the sphere at two points, giving two solutions.
- *Singular solution* - circle tangent to sphere resulting in one solution.
- *Singular solution* - circle lies on sphere -- physically unrealistic case!
- *No solution* - circle and sphere do not intersect

3.10 Kinematics Summary for Parallel Robots

Both the forward and inverse kinematics can pose difficult solutions. It is helpful to understand the geometry of motion, because this provides insights into the kinematics solutions. Grubler's Criterion does not readily apply to this class of complex mechanisms.