Adaptive Control of Nonlinearly Parameterized Systems: The Smooth Feedback Case

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***Abstract*—This paper studies global adaptive control of *nonlinearly parameterized* systems with *uncontrollable linearization*. Using a new parameter separation technique and the tool of adding a power integrator, we develop a feedback domination design approach for the explicit construction of a smooth adaptive controller that solves the problem of global state regulation. In contrast to the existing results in the literature, a key feature of our adaptive regulator is its minimum-order property, namely, no matter how big the number of unknown parameters is, the order of the dynamic compensator is identical to one, and is therefore minimal. As a consequence, global state regulation of feedback linearizable systems with nonlinear parameterization is achieved by one-dimensional adaptive controllers, without imposing any extra (e.g., convex/concave) conditions on the unknown parameters.**

***Index Terms*—Adding a power integrator, global adaptive stabilization, nonlinear parameterization, nonlinear systems with uncontrollable linearization, smooth feedback.**

I. Introduction

ADAPTIVE control of nonlinear systems with parametric uncertainty has been one of the active subjects in the field of nonlinear control. Two recent books, [14] and [24], provide a comprehensive report on the major developments in the area of adaptive control of feedback linearizable systems with *linear parameterization*. By comparison, little progress has been made for adaptive control of *nonlinearly parameterized* systems involving inherent nonlinearity, in the sense that the system may be neither feedback linearizable nor affine in the control input, and its *linearization is uncontrollable*. As a matter of fact, even in the case of feedback linearizable systems with nonlinear parameterization, very few results are available in the literature and global adaptive regulation has remained largely open for more than a decade.

As demonstrated in [1], [23], [2], and [4], nonlinear parameterization can be found in various practical control problems. For instance, it arises naturally in physical systems such as biochemical processes [4] and machines with friction [1]. Dealing with this type of *nonlinearly parameterized* dynamic systems is not only interesting theoretically (as it represents a challenge for adaptive control), but also important from a viewpoint of practical applications. In the past few years, several researchers started working on this difficult problem and obtained some interesting results [23], [2], [16], [17], [4]. It must be noticed, however, that most of the results were derived under various conditions imposed on the unknown parameters. One of common assumptions is that *bound of the nonlinear parameters is known*. Under such a condition, the problem of global adaptive control by output feedback was solved for nonlinearly parameterized systems [23]. The other condition is the so-called *convex/concave* parameterization which has been assumed in [2], [16], and [17], where a min–max strategy was proposed for the design of adaptive tracking controllers. However, without imposing any condition on the parameters, global adaptive regulation of *nonlinearly parameterized* systems has been recognized as a challenging open problem, particularly in the case of nonlinear systems with *uncontrollable linearization*.

To address this difficult issue, new nonlinear adaptive control strategies must be developed. This is because most of the adaptive control schemes [14], [24], [25], and [18], on one hand, rely heavily on *linear parameterization*, and on the other hand, they are only applicable to *feedback linearizable* systems in a triangular form [14], [24], [25], [18], [17]. The feedback linearizable condition was relaxed in [20], where a solution to the problem of global adaptive regulation was given for linearly parameterized, high-order systems. The progress was made possible due to the development of a novel feedback design technique called *adding a power integrator*, which was motivated by homogeneous feedback stabilization [3], [8], [9], [12], [13], [6], [7] and proposed initially in [19] for global stabilization of nonlinear systems with uncontrollable linearization. It turns out that this technique is also useful in solving adaptive regulation of high-order systems [20]. The essential idea behind adding a power integrator is that the feedback domination design, instead of feedback cancellation, is employed to deal with the nonlinearities of the system. While the backstepping design [14], [24] is only applicable to feedback linearizable systems, the adding a power integrator technique [19], [20] appears to be extremely powerful in dealing with a class of inherently nonlinear systems with uncontrollable linearization.

On the other hand, to deal with *nonlinear parameterization*, one must devise an innovative way to overcome the obstacle caused by the unknown nonlinear parameters which are exceptionally difficult to estimate. Of course, one way is to find a transformation which transforms nonlinearly parameterized systems into systems with a linear parameterization. However,

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this is not a trivial task and there is no systematic method available currently. Another possible method is to impose certain conditions on the system parameters such as convex/concave parameterization, as done in [2] and [17].

In this paper, we present a new approach to deal with general nonlinear parameterizations. First, we show that for every continuous function with nonlinear parameters, it is always possible to "separate" the nonlinear parameters from the nonlinear function. Then, in order to effectively use this parameter separation technique for the design of adaptive control systems, we modify the tool of adding a power integrator accordingly. A key feature of the adding a power integrator technique [19], [20] is that it only requires the knowledge of the upper bound of nonlinearities whose exact information needs not to be known. By taking such advantage, together with the novel parameter separation technique, we are able to remove the linear parameterization condition, and in turn solve the open problem of global adaptive stabilization for a class of *nonlinearly parameterized* systems with *uncontrollable linearization*. A systematic design procedure is given for the explicit construction of smooth, one-dimensional (1-D) adaptive controllers which achieve asymptotic state regulation with global stability. As a consequence, we arrive at an important conclusion on global adaptive stabilization of affine systems with nonlinear parameterization: every feedback linearizable or minimum-phase system with nonlinear parameterization is globally stabilizable by a smooth 1-D adaptive controller, without imposing any extra condition such as convex/concave condition [2], [16], [17] on the unknown parameters.

It is worth emphasizing that the uncertain systems considered in the paper are inherently nonlinear in the sense that: 1) the parameters appear nonlinearly and belong to an *unknown* compact set, i.e., no prior knowledge is required on the bound of the unknown parameters; 2) the systems are high-order because the Jacobian linearization is null or uncontrollable; and 3) the systems are *not necessarily in a lower triangular form*. Therefore, the class of systems is much more general than (also significant different from) feedback linearizable systems with linear parameterization [14], [24], [25] and must be dealt with by pure nonlinear methods, i.e., no feedback linearization design works, even locally.

II. Preliminary

A standard adaptive global stabilization problem or, what is the same, the problem of adaptive regulation with global stability is formulated as follows: for a smooth nonlinear system

$$\dot{\zeta} = f(\zeta, u, \theta) \tag{2.1}$$

with an unknown parameter vector θ, find, if possible, a smooth adaptive controller

$$\begin{aligned} \dot{\hat{\theta}} &= \psi(\zeta, \hat{\theta}) \quad \psi(0, 0) = 0 \\ u &= u(\zeta, \hat{\theta}) \quad u(0, 0) = 0 \end{aligned} \tag{2.2}$$

such that the closed-loop system (2.1) and (2.2) is globally stable in the sense of Lyapunov, and global asymptotic regulation of the state is achieved, i.e., $\lim_{t\to\infty} \zeta(t) = 0$.

Under the linear parameterization condition, global adaptive regulation has been investigated in a number of papers ([14], [24], [25], and [18]), where globally stabilizing smooth adaptive controllers of the form (2.2), with $\dim \hat{\theta} = \dim \theta$, were designed for the feedback linearizable system

$$\begin{aligned} \dot{z} &= f_0(z, x_1) + \phi_0(z, x_1)\theta \\ \dot{x}_1 &= x_2 + \phi_1(z, x_1)\theta \\ &\vdots \\ \dot{x}_r &= u + \phi_r(z, x_1, \ldots, x_r)\theta. \end{aligned} \tag{2.3}$$

By comparison, only few results in the literature addressed adaptive control of nonlinear systems with *nonlinear parameterization*, under conditions such as convex/concave parameterizations [23], [2], [4].

A longstanding open problem in the field of nonlinear adaptive control is the question of when global state regulation of *nonlinearly parameterized* systems can be solved by a smooth adaptive controller. In this paper, we address this challenging question and provide a partial solution to it. This is accomplished by characterizing sufficient conditions for the problem to be solvable for a class of high-order nonlinearly parameterized systems of the form

$$\begin{aligned} \dot{x}_1 &= d_1(x, u, \theta)x_2^{p_1} + \phi_1(x_1, x_2, \theta) \\ \dot{x}_2 &= d_2(x, u, \theta)x_3^{p_2} + \phi_2(x_1, x_2, x_3, \theta) \\ &\vdots \\ \dot{x}_n &= d_n(x, u, \theta)u^{p_n} + \phi_n(x_1, \ldots, x_n, u, \theta) \end{aligned} \tag{2.4}$$

where $u \in \mathbb{R}$ and $x \in \mathbb{R}^n$ are the system input and state, p_i, $i = 1, \ldots, n$ are *odd* positive integers, $\theta \in \mathbb{R}^s$ is an unknown constant vector, $d_i: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}$ and $\phi_i: \mathbb{R}^{i+1} \times \mathbb{R}^s \to \mathbb{R}$ are C^1 functions with $\phi_i(0, \ldots, 0, \theta) = 0$.

The controlled plant (2.4) represents a number of important classes of nonlinear systems with parametric uncertainty. The simplest case is the feedback linearizable system where $p_i = 1$, $\phi_i(x_1, \ldots, x_{i+1}, \theta) = \phi_i(x_1, \ldots, x_i, \theta)$, $d_i(\cdot) = 1$, $i = 1, \ldots, n$. The other interesting case of (2.4) is the class of high-order lower triangular systems with nonlinear parameterization. Finally, (2.4) encompasses a class of *nontriangular* systems with uncontrollable linearization (e.g., Example 5.3) that cannot be dealt with by existing methods.

In the rest of this section, we introduce two key lemmas which serve as a basis for the explicit construction of globally stabilizing smooth adaptive controllers for nonlinear systems (2.4). The first lemma provides a new parameter separation technique which enables one to deal with nonlinear parameterization. A successful combination of this lemma and the adding a power integrator technique [20] will result in a solution to the global adaptive regulation problem of nonlinearly parameterized systems (2.4).

Lemma 2.1: For any real-valued continuous function $f(x, y)$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, there are smooth scalar functions $a(x) \geq 0$, $b(y) \geq 0$, $c(x) \geq 1$ and $d(y) \geq 1$, such that

$$|f(x, y)| \leq a(x) + b(y) \tag{2.5}$$

$$|f(x, y)| \leq c(x)\, d(y). \tag{2.6}$$

Proof: For each $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, define

$$\Omega_x = \{(x, s) \,|\, s \in R^n, \, \|s\| \le \|x\|\}$$
$$\Omega_y = \{(t, y) \,|\, t \in R^m, \, \|t\| \le \|y\|\}$$

which are compact for every fixed (x, y).

When $\|y\| \le \|x\|$, the point (x, y) lies in the set Ω_x. As a consequence

$$|f(x, y)| \le A(x), \qquad A(x) := \max_{(t,s)\in\Omega_x} |f(t, s)|.$$

Similarly, it is easy to show that

$$|f(x, y)| \le B(y), \qquad B(y) := \max_{(t,s)\in\Omega_y} |f(t, s)|$$

when $\|x\| \le \|y\|$.

In view of the argument above, one concludes that for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$

$$|f(x, y)| \le A(x) + B(y).$$

By construction, the functions $A(x)$ and $B(y)$ are continuous, and, hence, can always be dominated by two smooth functions $a(x)$ and $b(y)$, respectively. Thus, (2.5) holds. Inequality (2.6) follows immediately from (2.5). In fact

$$|f(x, y)| \le a(x) + b(y) \le c(x)\, d(y)$$

where $c(x) := 1 + a(x) \ge 1$, $d(y) := 1 + b(y) \ge 1$. ∎

Example 2.2: Consider the smooth function $f(x, y) = e^{xy}$. By Lemma 2.1

$$A(x) := \max_{(t,s)\in\Omega_x} e^{ts} = \max_{|s|\le|x|} e^{xs} = e^{x^2}$$

which is smooth. Hence, $f(x, y) \le e^{x^2} + e^{y^2} \le (1+e^{x^2})(1+e^{y^2})$.

Example 2.3: For the continuous function $f(x, y) = |x|^{|y|}$, a straightforward calculation gives

$$A(x) := \max_{|s|\le|x|} |x|^{|s|} = \begin{cases} 1, & |x| < 1 \\ |x|^{|x|} & |x| \ge 1, \end{cases}$$
$$B(y) := \max_{|t|\le|y|} |t|^{|y|} = |y|^{|y|}.$$

Obviously

$$A(x) \le a(x) = (1 + x^2)^{(1+x^2)/4}$$
$$B(y) \le b(y) = (1 + y^2)^{(1+y^2)/4}.$$

Thus, one can choose the C^∞ functions $c(x) = a(x)$ and $d(y) = b(y)$, such that (2.5) and (2.6) hold.

The following Lemma is a consequence of Young's inequality and plays a key role in the adding a power integrator design.

Lemma 2.4 [20]: For any positive integers m, n and any real-valued function $\pi(x, y) > 0$,

$$|x|^m|y|^n \le \frac{m}{m+n}\,\pi(x, y)|x|^{m+n} + \frac{n}{m+n}\,\pi^{-(m/n)}(x, y)|y|^{m+n}. \qquad (2.7)$$

III. Triangular Systems With Nonlinear Parameterization

With the aid of Lemmas 2.1 and 2.4, we can present a feedback domination design approach which leads to solutions to the problem of adaptive regulation with global stability, for two important classes of nonlinearly parameterized systems in a lower triangular form.

A. High-Order Nonlinear Systems With Uncontrollable Linearization

For the sake of simplicity, we first consider the nonlinearly parameterized system (2.4) with $\phi_i(x_1, \ldots, x_{i+1}, \theta) \equiv \phi_i(x_1, \ldots, x_i, \theta)$, which represents an important class of high-order lower triangular systems, i.e.,

$$\begin{aligned} \dot{x}_1 &= x_2^{p_1} + \phi_1(x_1, \theta) \\ &\vdots \\ \dot{x}_{n-1} &= x_n^{p_{n-1}} + \phi_{n-1}(x_1, \ldots, x_{n-1}, \theta) \\ \dot{x}_n &= u^{p_n} + \phi_n(x_1, \ldots, x_n, \theta). \end{aligned} \qquad (3.1)$$

It has been known that even under the linear parameterization condition, global adaptive regulation of the high-order triangular system (3.1) is a nontrivial problem, due to the *lack of feedback linearizability and affiness*. As a matter of fact, counterexamples given in [19] have indicated that without imposing suitable growth conditions on p_i and $\phi_i(\cdot)$, the problem is usually unsolvable by any *smooth* adaptive controller. In the case of *nonlinearly parameterized* systems (3.1), the following assumptions which can be viewed as a *high-order version of feedback linearizable condition* are needed in order to solve the adaptive control problem.

Assumption 3.1: $p_1 \ge p_2 \ge \cdots \ge p_n \ge 1$ are odd integers.

Assumption 3.2: For $i = 1, \ldots, n$

$$|\phi_i(x_1, \ldots, x_i, \theta)| \le (|x_1|^{p_i} + \cdots + |x_i|^{p_i})b_i(x_1, \ldots, x_i, \theta) \qquad (3.2)$$

where $b_i(\cdot)$ is a nonnegative continuous function.

Remark 3.3: By Lemma 2.1, there exist two smooth functions $c_i(\theta) \ge 1$ and $\gamma_i(x_1, \ldots, x_i) \ge 1$ satisfying

$$b_i(x_1, \ldots, x_i, \theta) \le \gamma_i(x_1, \ldots, x_i)c_i(\theta).$$

Since θ is a constant, $c_i(\theta)$ is a constant as well. Let $\Theta := \sum_{i=1}^n c_i(\theta)$ be a new unknown constant. Then, Assumption 3.2 implies that there are smooth functions $\gamma_i(x_1, \ldots, x_i) \ge 1$ and an unknown constant $\Theta \ge 1$, such that

$$|\phi_i(x_1, \ldots, x_i, \theta)| \le (|x_1|^{p_i} + \cdots + |x_i|^{p_i})\gamma_i(x_1, \ldots, x_i)\Theta. \qquad (3.3)$$

Lemma 2.1, together with Remark 3.3, provides a new way to deal with the nonlinear parameterization problem. In this paper, in lieu of estimating the unknown parameter $\theta \in \mathbb{R}^s$, we shall estimate the unknown constant Θ which is scalar and positive. However, due to the fact that in (3.3) Θ only appears linearly in the bounding function $\gamma_i(\cdot)$, there is a technical difficulty in processing an adaptive control design. Namely, in order to take advantage of the linear-like parameterization condition (3.3), only the bounding function $\gamma_i(\cdot)$, instead of $\phi_i(\cdot)$, can be used in the design of adaptive controllers. To overcome this major difficulty, we propose a feedback domination design approach. In contrast to the existing adaptive control schemes for linearly parameterized systems such as (2.3) [which are based on feedback cancellation and usually require the precise information of $\phi_i(\cdot)$], our new feedback domination design needs not to know the precise information of $\phi_i(\cdot)$ but $\gamma_i(\cdot)$, and therefore leads to a solution to global adaptive regulation of nonlinearly parameterized systems (3.1).

Theorem 3.4: Under Assumptions 3.1 and 3.2, there is a *1-D* smooth adaptive controller

$$\begin{aligned}\dot{\hat{\Theta}} &= \psi(x_1, \ldots, x_n, \hat{\Theta}), \qquad \hat{\Theta} \in \mathbb{R} \\ u &= u(x_1, \ldots, x_n, \hat{\Theta})\end{aligned} \tag{3.4}$$

such that the closed-loop system (3.1)–(3.4) is globally stable in the sense of Lyapunov. Moreover, global asymptotic regulation of the state is achieved, i.e.,

$$\lim_{t\to\infty} x(t) = 0 \qquad \forall\,(x(0), \hat{\Theta}(0)) \in \mathbb{R}^n \times \mathbb{R}.$$

Proof: The proof is based on a feedback domination design approach which combines the technique of *adding one power integrator* [20] with the new parameter separation method (i.e., Lemma 2.1 and Remark 3.3). Using the feedback domination design, we explicitly construct a control Lyapunov function and a minimum-order adaptive controller of the form (3.4) that solves the problem.

Initial Step: Let $\Theta := \sum_{i=1}^{n} c_i(\theta)$ be the unknown constant defined in Remark 3.3. Define $\tilde{\Theta}(t) = \Theta - \hat{\Theta}(t)$, where $\hat{\Theta}(t)$ is the estimate of Θ to be designed later. Consider the Lyapunov function $V_1(x_1, \tilde{\Theta}) = (1/2)x_1^2 + (1/2)\tilde{\Theta}^2$. By *A3.2* and Remark 3.3, it is easy to show that

$$\dot{V}_1(x_1, \tilde{\Theta}) \le x_1 x_2^{p_1} + x_1^{p_1+1}\gamma_1(x_1)(\tilde{\Theta} + \hat{\Theta}) - \dot{\hat{\Theta}}(t)\tilde{\Theta}(t).$$

With the choice of the smooth virtual controller

$$x_2^* = -x_1\left(n + \gamma_1(x_1)\sqrt{\hat{\Theta}^2 + 1}\right)^{1/p_1} := -x_1\beta_1(x_1, \hat{\Theta})$$

we have

$$\begin{aligned}\dot{V}_1(x_1, \tilde{\Theta}) \le{}& -n x_1^{p_1+1} + x_1(x_2^{p_1} - x_2^{*p_1}) \\ &+ \left(\Psi_1(x_1) - \dot{\hat{\Theta}}(t)\right)(\tilde{\Theta}(t) + \eta_1)\end{aligned} \tag{3.5}$$

where $\Psi_1(x_1) = x_1^{p_1+1}\gamma_1(x_1) \ge 0$ and $\eta_1 = 0$.

Inductive Step: Suppose for the system (3.1) with dimension k, there are a set of smooth virtual controllers $x_1^*, \ldots, x_{k+1}^*$, defined by

$$\begin{aligned} x_1^* &= 0 & \xi_1 &= x_1 - x_1^* \\ x_2^* &= -\xi_1\beta_1(x_1, \hat{\Theta}) & \xi_2 &= x_2 - x_2^* \\ &\vdots & &\vdots \\ x_{k+1}^* &= -\xi_k\beta_k(x_1, \ldots, x_k, \hat{\Theta}) & \xi_{k+1} &= x_{k+1} - x_{k+1}^* \end{aligned} \tag{3.6}$$

with $\beta_1(x_1, \hat{\Theta}) > 0, \ldots, \beta_k(x_1, \ldots, x_k, \hat{\Theta}) > 0$ being *smooth*, such that

$$\begin{aligned}&\dot{V}_k(\xi_1, \ldots, \xi_k, \tilde{\Theta})\Big|_{(3.1)\text{–}(3.6)} \\ &\le -(n-k+1)(\xi_1^{p_1+1} + \cdots + \xi_k^{p_1+1}) \\ &\quad + \xi_k^{p_1-p_k+1}(x_{k+1}^{p_k} - x_{k+1}^{*p_k}) \\ &\quad + \left(\Psi_k(\xi_1, \ldots, \xi_k, \hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\xi_1, \ldots, \xi_k, \hat{\Theta})\right)\end{aligned} \tag{3.7}$$

where

$$V_k(\xi_1, \ldots, \xi_k, \tilde{\Theta}) := \sum_{j=1}^{k} \frac{\xi_j^{p_1-p_j+2}}{p_1 - p_j + 2} + \frac{\tilde{\Theta}^2}{2}$$

is positive definite and proper. Moreover

$$\begin{aligned}0 &\le \Psi_k(\xi_1, \ldots, \xi_k, \hat{\Theta}) \\ &\le (\xi_1^{p_1+1} + \cdots + \xi_k^{p_1+1})\alpha_k(\xi_1, \ldots, \xi_k, \hat{\Theta}) \\ &\quad \text{for } C^\infty \alpha_k(\cdot) \ge 0.\end{aligned} \tag{3.8}$$

Then, when the dimension of (3.1) is equal to $k+1$, we claim that (3.7) and (3.8) also hold. To see why this is the case, consider the Lyapunov function

$$V_{k+1}(\xi_1, \ldots, \xi_{k+1}, \tilde{\Theta}) = V_k(\xi_1, \ldots, \xi_k, \tilde{\Theta}) + \frac{\xi_{k+1}^{p_1-p_{k+1}+2}}{p_1 - p_{k+1} + 2}.$$

Clearly

$$\begin{aligned}\dot{V}_{k+1} \le{}& -(n-k+1)(\xi_1^{p_1+1} + \cdots + \xi_k^{p_1+1}) \\ &+ \xi_k^{p_1-p_k+1}(x_{k+1}^{p_k} - x_{k+1}^{*p_k}) \\ &+ \left(\Psi_k(\cdot) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\cdot)\right) + \xi_{k+1}^{p_1-p_{k+1}+1} \\ &\cdot \left[x_{k+2}^{p_{k+1}} + \phi_{k+1}(\cdot) - \sum_{j=1}^{k} \frac{\partial x_{k+1}^*}{\partial x_j}\dot{x}_j - \frac{\partial x_{k+1}^*}{\partial \hat{\Theta}}\dot{\hat{\Theta}}\right].\end{aligned} \tag{3.9}$$

By Assumption 3.2 and Remark 3.3

$$\begin{aligned}&\left|\phi_{k+1}(\cdot) - \sum_{j=1}^{k} \frac{\partial x_{k+1}^*}{\partial x_j}\dot{x}_j\right| \\ &\le \sum_{l=1}^{k+1} |x_l|^{p_{k+1}}\gamma_{k+1}(x_1, \ldots, x_{k+1})\Theta + \sum_{j=1}^{k}\left|\frac{\partial x_{k+1}^*}{\partial x_j}\right| \\ &\quad \cdot \left(|x_{j+1}|^{p_j} + \sum_{l=1}^{j} |x_l|^{p_j}\gamma_j(x_1, \ldots, x_j)\Theta\right).\end{aligned}$$

Since $\Theta \ge 1$ and $p_1 \ge p_2 \cdots \ge p_{k+1}$, there is a smooth function $\rho_{k+1}(x_1, \ldots, x_{k+1}, \hat{\Theta}) \ge 0$, such that

$$\begin{aligned}\left|\phi_{k+1}(\cdot) - \sum_{j=1}^{k} \frac{\partial x_{k+1}^*}{\partial x_j}\dot{x}_j\right| \le{}& \sum_{l=1}^{k+1} \\ &\cdot |x_l|^{p_{k+1}}\rho_{k+1}(x_1, \ldots, x_{k+1}, \hat{\Theta})\Theta.\end{aligned} \tag{3.10}$$

This, together with (3.6), implies

$$\left|\phi_{k+1}(\cdot) - \sum_{j=1}^{k} \frac{\partial x_{k+1}^*}{\partial x_j}\,\dot{x}_j\right| \le \sum_{l=1}^{k+1} \cdot|\xi_l|^{p_{k+1}}\omega_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta})\Theta \quad (3.11)$$

where $\omega_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta}) \ge 0$ is a smooth function.

Using Lemma 2.4 and (3.10), (3.11), it is deduced that there is a smooth function $\overline{\omega}_{k+1}(\cdot) \ge 0$ satisfying

$$\begin{aligned}
&\left|\xi_{k+1}^{p_1-p_{k+1}+1}\right|\left|\phi_{k+1}(\cdot) - \sum_{j=1}^{k} \frac{\partial x_{k+1}^*}{\partial x_j}\,\dot{x}_j\right| \\
&\le \left[\frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{2(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} + \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta})\right]\Theta \\
&\le \left[\frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{2(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} + \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\cdot)\right]\tilde{\Theta} \\
&\quad + \frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{4} + \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\cdot)\sqrt{\hat{\Theta}^2+1}.
\end{aligned} \quad (3.12)$$

Recall that for any odd integer $p \ge 1$

$$\begin{aligned}
|(a+b)^p - b^p| &\le p|a|((a+b)^{p-1} + b^{p-1}) \\
&\le p|a|[2^{p-2}(a^{p-1}+b^{p-1}) + b^{p-1}].
\end{aligned}$$

With this in mind, we have

$$\begin{aligned}
&\left|\xi_k^{p_1-p_k+1}[(\xi_{k+1}+x_{k+1}^*)^{p_k} - x_{k+1}^{*p_k}]\right| \\
&\le p_k\left[2^{p_k-2}\left|\xi_{k+1}^{p_k}\xi_k^{p_1-p_k+1}\right| + (1+2^{p_k-2})\beta_k^{p_k}(\cdot)\left|\xi_k^{p_1}\xi_{k+1}\right|\right] \\
&\le \tfrac{1}{4}\,(\xi_1^{p_1+1}+\cdots+\xi_k^{p_1+1}) + \xi_{k+1}^{p_1+1}\tilde{\gamma}_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta})
\end{aligned} \quad (3.13)$$

for a smooth function $\tilde{\gamma}(\cdot) \ge 0$. The last inequality follows from Lemma 2.4.

Substituting (3.12) and (3.13) into (3.9) gives

$$\begin{aligned}
\dot{V}_{k+1} \le &-\left(n-k+\tfrac{1}{2}\right)(\xi_1^{p_1+1}+\cdots+\xi_k^{p_1+1}) \\
&+ \xi_{k+1}^{p_1-p_{k+1}+1}x_{k+2}^{p_{k+1}} + \xi_{k+1}^{p_1+1} \\
&\cdot\left[\overline{\omega}_{k+1}(\cdot)\sqrt{\hat{\Theta}^2+1} + \tilde{\gamma}_{k+1}(\cdot)\right] \\
&+ \left[\frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{2(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} + \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\cdot)\right]\tilde{\Theta} \\
&- \xi_{k+1}^{p_1-p_{k+1}+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\,\dot{\hat{\Theta}} + \left(\Psi_k(\cdot) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\cdot)\right).
\end{aligned} \quad (3.14)$$

Define

$$\begin{aligned}
&\Psi_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta}) \\
&\quad = \Psi_k(\xi_1,\,\ldots,\,\xi_k,\,\hat{\Theta}) + \frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{2(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} \\
&\qquad + \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\cdot) \\
&\eta_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta}) \\
&\quad = \eta_k(\xi_1,\,\ldots,\,\xi_k,\,\hat{\Theta}) + \xi_{k+1}^{p_1-p_{k+1}+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}.
\end{aligned}$$

Using (3.8), it is not difficult to verify that

$$\begin{aligned}
0 &\le \Psi_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta}) \\
&\le (\xi_1^{p_1+1}+\cdots+\xi_{k+1}^{p_1+1})\alpha_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta}) \\
&\quad \text{for a } C^\infty\ \alpha_{k+1}(\cdot) \ge 0.
\end{aligned} \quad (3.15)$$

Moreover, (3.14) can be rewritten as follows:

$$\begin{aligned}
\dot{V}_{k+1} \le &-\left(n-k+\tfrac{1}{2}\right)(\xi_1^{p_1+1}+\cdots+\xi_k^{p_1+1}) \\
&+ \xi_{k+1}^{p_1-p_{k+1}+1}x_{k+2}^{p_{k+1}} + \xi_{k+1}^{p_1+1} \\
&\cdot\left[\overline{\omega}_{k+1}(\cdot)\sqrt{\hat{\Theta}^2+1} + \tilde{\gamma}_{k+1}(\cdot)\right] \\
&+ \left(\Psi_{k+1}(\cdot) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_{k+1}(\cdot)\right) \\
&- \left[\frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{2(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} + \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\cdot)\right]\eta_k(\cdot) \\
&- \xi_{k+1}^{p_1-p_{k+1}+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\,\Psi_{k+1}(\cdot).
\end{aligned} \quad (3.16)$$

By (3.15), we have

$$\begin{aligned}
&\left|\left(\frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{2(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} + \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\cdot)\right)\eta_k(\cdot) + \xi_{k+1}^{p_1-p_{k+1}+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\,\Psi_{k+1}(\cdot)\right| \\
&\le \tfrac{1}{4}\sum_{l=1}^{k}\xi_l^{p_1+1} + \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\cdot)\sqrt{\eta_k^2(\cdot)+1} \\
&\quad + \left|\xi_{k+1}^{p_1-p_{k+1}+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\right|(\xi_1^{p_1+1}+\cdots+\xi_{k+1}^{p_1+1})\alpha_{k+1}(\cdot) \\
&\le \tfrac{1}{2}\,(\xi_1^{p_1+1}+\cdots+\xi_k^{p_1+1}) + \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\cdot)\sqrt{\eta_k^2(\cdot)+1} \\
&\quad + \xi_{k+1}^{p_1+1}\sum_{j=1}^{k+1}\tilde{\beta}_{(k+1),\,j}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta}).
\end{aligned} \quad (3.17)$$

The last inequality follows from the following relation (which is a consequence of Lemma 2.4):

$$\left|\xi_{k+1}^{p_1+1-p_{k+1}}\right| \xi_j^{p_1+1} \left|\frac{\partial x_{k+1}^*}{\partial \hat{\Theta}} \alpha_{k+1}(\cdot)\right| \le \frac{\xi_j^{p_1+1}}{4} + \xi_{k+1}^{p_1+1} \tilde{\beta}_{k+1,j}(\xi_1, \ldots, \xi_{k+1}, \hat{\Theta}), \qquad 1 \le j \le k$$

where $\tilde{\beta}_{k+1,j}(\cdot) \ge 0$, $1 \le j \le k$, are smooth functions and $\tilde{\beta}_{k+1,k+1}(\cdot) \ge |\xi_{k+1}|^{p_1-p_{k+1}+1} \, |\partial x_{k+1}^*/\partial\hat{\Theta}|\alpha_{k+1}(\cdot)$ is a smooth function.

Putting (3.17) and (3.16) together, one arrives at

$$\begin{aligned}\dot{V}_{k+1}(\cdot) \le& -(n-k)(\xi_1^{p_1+1} + \cdots + \xi_k^{p_1+1}) + \xi_{k+1}^{p_1-p_{k+1}+1} x_{k+2}^{p_{k+1}} \\ &+ \xi_{k+1}^{p_1+1}\left[\overline{\omega}_{k+1}(\cdot)\sqrt{\hat{\Theta}^2+1} + \tilde{\gamma}_{k+1}(\cdot) + \hat{\gamma}_{k+1}(\cdot)\right] \\ &+ \left(\Psi_{k+1}(\cdot) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_{k+1}(\cdot)\right) \end{aligned} \tag{3.18}$$

where $\hat{\gamma}_{k+1}(\cdot) = \overline{\omega}_{k+1}(\cdot)\sqrt{\eta_k^2(\cdot)+1} + \sum_{j=1}^{k+1} \tilde{\beta}_{k+1,j}(\xi_1, \ldots, \xi_{k+1}, \hat{\Theta})$.

Now, it is easy to see that the smooth virtual controller

$$\begin{aligned} x_{k+2}^* &= -\xi_{k+1}\left[n-k+\rho_{k+1}(\xi_1, \ldots, \xi_{k+1}, \hat{\Theta})\right]^{1/p_{k+1}} \\ \rho_{k+1}(\cdot) &= \overline{\omega}_{k+1}(\cdot)\sqrt{\hat{\Theta}^2+1} + \tilde{\gamma}_{k+1}(\cdot) + \hat{\gamma}_{k+1}(\cdot) \ge 0 \end{aligned} \tag{3.19}$$

renders

$$\begin{aligned} &\dot{V}_{k+1}(\xi_1, \ldots, \xi_{k+1}, \tilde{\Theta}) \\ &\quad \le -(n-k)(\xi_1^{p_1+1} + \cdots + \xi_{k+1}^{p_1+1}) \\ &\qquad + \xi_{k+1}^{p_1-p_{k+1}+1}(x_{k+2}^{p_{k+1}} - x_{k+2}^{*p_{k+1}}) \\ &\qquad + \left(\Psi_{k+1}(\xi_1, \ldots, \xi_{k+1}, \hat{\Theta}) - \dot{\hat{\Theta}}\right) \\ &\qquad \cdot \left(\tilde{\Theta} + \eta_{k+1}(\xi_1, \ldots, \xi_{k+1}, \hat{\Theta})\right). \end{aligned} \tag{3.20}$$

The aforementioned inductive argument shows that (3.7) holds for $k = n$. In fact, at the nth step, one can construct explicitly a global change of coordinates $(\xi_1, \ldots, \xi_n)$, a positive–definite and proper Lyapunov function $V_n(\xi_1, \ldots, \xi_n, \tilde{\Theta})$ and a smooth controller $u^*(\xi_1, \ldots, \xi_n, \hat{\Theta})$ of the form (3.19), such that

$$\begin{aligned} &\dot{V}_n(\xi_1, \ldots, \xi_n, \tilde{\Theta})\Big|_{(3.1)-(3.6)} \\ &\quad \le -(\xi_1^{p_1+1} + \cdots + \xi_n^{p_1+1}) + \xi_n^{p_1-p_n+1}(u^{p_n} - u^{*p_n}) \\ &\qquad + \left(\Psi_n(\xi_1, \ldots, \xi_n, \hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_n(\xi_1, \ldots, \xi_n, \hat{\Theta})\right). \end{aligned} \tag{3.21}$$

Therefore, the 1-D smooth adaptive controller

$$\begin{aligned} \dot{\hat{\Theta}} &= \Psi_n(\xi_1, \ldots, \xi_n, \hat{\Theta}) \\ u &= u^*(\xi_1, \ldots, \xi_n, \hat{\Theta}) \end{aligned} \tag{3.22}$$

is such that

$$\dot{V}_n(\xi_1, \ldots, \xi_n, \tilde{\Theta})\Big|_{(3.1)-(3.22)} \le -(\xi_1^{p_1+1} + \cdots + \xi_n^{p_1+1}). \tag{3.23}$$

In view of the classical Lyapunov stability theory, we conclude that the closed-loop system is globally stable at the equilibrium $(\xi_1, \ldots, \xi_n, \tilde{\Theta}) = (0, \ldots, 0)$. Moreover, by La Salle's invariance principle all the bounded trajectories of the closed-loop system approach the largest invariant set contained in $\{(\xi_1, \ldots, \xi_n, \tilde{\Theta})\colon \dot{V}_n = 0\}$. Hence, $\lim_{t\to\infty}(\xi_1(t), \ldots, \xi_n(t))^T = 0$. This, together with the relation (3.6) (with $k = n$), implies

$$\lim_{t\to\infty}(x_1(t), \ldots, x_n(t))^T = 0 \qquad \forall (x(0), \hat{\Theta}(0)) \in \mathbb{R}^n \times \mathbb{R}.$$

∎

It is clear from the proof of Theorem 3.4 that $\theta \in \mathbb{R}^s$ needs not be a *constant* vector. In fact, θ can be a C^0 time-varying function as long as $\theta(t)$ is bounded, although *its bound may be unknown*. In other words, the adaptive control problem is still solvable for the *time-varying* nonlinearly parameterized system (3.1), with $\theta\colon \mathbb{R} \to \mathbb{R}^s$ being a continuous function of t, bounded by an *unknown constant* $\overline{\theta}$.

Corollary 3.5: For the nonlinearly parameterized system (3.1) with $\theta = \theta(t)$ being a C^0 time-varying signal whose bound is an unknown constant, there is a 1-D C^∞ adaptive controller of the form (3.4) such that the closed-loop system is globally stable and $\lim_{t\to\infty} x(t) = 0$, if Assumptions 3.1 and 3.2 hold. ∎

From now on, we shall only deal with, without loss of generality, the unknown constant vector $\theta \in \mathbb{R}^s$ rather than the unknown time-varying signal $\theta(t)$. However, all the adaptive control results presented in the remainder of this paper can also be applied, as illustrated by Corollary 3.5, to the corresponding nonlinearly parameterized systems with an unknown bounded time-varying signal, under appropriate conditions.

In the case of linearly parameterized systems, Theorem 3.4 has the following corollary which refines the adaptive control result obtained in [20].

Corollary 3.6: Consider the high-order system (3.1) in which $\phi_i(x_1, \ldots, x_i, \theta) = \phi_i(x_1, \ldots, x_i)\theta$. If Assumtpion 3.1 holds and

$$\|\phi_i(x_1, \ldots, x_i)\| \le (|x_1|^{p_i} + \cdots + |x_i|^{p_i})\,\gamma_i(x_1, \ldots, x_i) \qquad i = 1, \ldots, n$$

then global adaptive regulation of (3.1) is solvable by the 1-D C^∞ adaptive controller (3.4). ∎

Corollary 3.6 indicates that global adaptive stabilization of systems (3.1) with linear parameterization is achievable by a smooth *1-D* (rather than s-dimensional [20]) adaptive controller. However, the feedback design methods in [20] and this paper are substantially different. Indeed, the technique in [20] can only be used to deal with triangular systems with linear parameterization, and is by no means applicable to the nonlinearly parameterized case.

Remark 3.7: It is worth pointing out that the two control schemes also result in dramatically different adaptive controllers. As a matter of fact, for the high-order, linearly parameterized system (3.1) with an s-dimensional unknown parameter θ, the adaptive controller obtained in [20] is an s-dimensional dynamic state compensator which has been viewed as the simplest adaptive controller in the literature, because the order of the adaptive compensator is equal to the number of unknown parameters. However, using our new feedback domination design method, it is possible to construct a smooth, *1-D* adaptive controller that achieves global state regulation, no matter how big the number of unknown parameters is. In other words, a significant feature of the new adaptive regulator presented in Theorem 3.4 is its *minimum-order* property. That is, the order of the dynamic compensator is equal to one and, hence, is minimal.

B. High-Order Cascade Systems

In this subsection, we briefly discuss how the adaptive stabilization result obtained for triangular systems can be extended to the following class of cascade systems with nonlinear parameterization:

$$\begin{aligned}\dot{z} &= f(z, x_1, \theta)\\ \dot{x}_1 &= x_2^{p_1} + \phi_1(z, x_1, \theta)\\ &\vdots\\ \dot{x}_{r-1} &= x_r^{p_{r-1}} + \phi_{r-1}(z, x_1, \dots, x_{r-1}, \theta)\\ \dot{x}_r &= u^{p_r} + \phi_r(z, x_1, \dots, x_r, \theta)\end{aligned} \tag{3.24}$$

where $(z, x_1, \dots, x_r) \in R^n$ represents the system state, $u \in \mathbb{R}$ is the control input and $\theta \in R^s$ is an unknown constant vector. The functions $f(\cdot)$ and $\phi_i(\cdot), i = 1, \dots, r$, are assumed to be smooth, vanishing at the origin $(z, x) = (0, 0)$.

The following assumptions are a modified version of Assumptions 3.1 and 3.2.

Assumption 3.8: $p_1 \geq \cdots \geq p_r \geq 1$ are odd integers.

Assumption 3.9: There are continuous functions $b_0(z, x_1, \theta) \geq 0$ and $b_i(z, x_1, \dots, x_i, \theta) \geq 0$, such that

$$\begin{aligned}&\|f(z, x_1, \theta)\|\\ &\quad \leq (\|z\|^{p_1} + |x_1|^{p_1})\, b_0(z, x_1, \theta)\end{aligned} \tag{3.25}$$

$$\begin{aligned}&|\phi_i(z, x_1, \dots, x_i, \theta)|\\ &\quad \leq (\|z\|^{p_i} + |x_1|^{p_i} + \cdots + |x_i|^{p_i})\\ &\qquad \cdot b_i(z, x_1, \dots, x_i, \theta), \qquad i = 1, \dots, r.\end{aligned} \tag{3.26}$$

Theorem 3.10: Suppose there are a smooth Lyapunov function $U(z)$, which is positive–definite and proper, and a smooth function $\alpha(z)$ with $\alpha(0) = 0$, such that

$$\frac{\partial U}{\partial z} f(z, x_1, \theta) \leq -\|z\|^{p_0+1} + |x_1 - \alpha(z)|\varphi(z, x_1, \theta) \tag{3.27}$$

where $p_0 \geq p_1$ is an odd integer, $\varphi(\cdot)$ is continuous and $0 \leq \varphi(z, x_1, \theta) \leq (\|z\|^{p_0} + |x_1|^{p_0})h(z, x_1, \theta)$ for a C^0 function $h(z, x_1, \theta) \geq 0$. Then, under Assumptions 3.8 and 3.9, there exists a smooth, 1-D adaptive controller (3.4) that solves the global adaptive regulation problem of (3.24).

Proof: The proof is similar to that of Theorem 3.4. The only difference is that at Step 1, we choose $x_1^* = \alpha(z)$ instead of $x_1^* = 0$. For convenience, we give the first step of the proof.

Let $\xi_1 = x_1 - \alpha(z)$ and $\tilde{\Theta} = \Theta - \hat{\Theta}$. Consider the Lyapunov function

$$V_1(z, x_1, \tilde{\Theta}) = (r+1)U(z) + \frac{\xi_1^{p_0-p_1+2}}{p_0 - p_1 + 2} + \frac{\tilde{\Theta}^2}{2}$$

which is positive–definite and proper. Then

$$\begin{aligned}\dot{V}_1 \leq &-(r+1)\|z\|^{p_0+1}\\ &+ (r+1)|\xi_1|(\|z\|^{p_0} + |x_1|^{p_0})h(z, x_1, \theta) + \xi_1^{p_0-p_1+1}\\ &\cdot\left(x_2^{p_1} + \phi_1(z, x_1, \theta) - \frac{\partial \alpha(z)}{\partial z} f(z, x_1, \theta)\right) - \dot{\hat{\Theta}}\tilde{\Theta}.\end{aligned}$$

Using Lemma 2.1, Assumption 3.9, and the fact that $p_0 \geq p_1$, it is not difficult to show that there are smooth functions $\gamma_0(z, x_1)$ and $\tilde{\gamma}_0(z, x_1)$ satisfying

$$\begin{aligned}\dot{V}_1 \leq &-(r+1)\|z\|^{p_0+1} + |\xi_1|(\|z\|^{p_0} + |\xi_1|^{p_0})\gamma_0(z, x_1)\Theta\\ &+ \xi_1^{p_0-p_1+1}x_2^{p_1}\\ &+ |\xi_1|^{p_0-p_1+1}(\|z\|^{p_1} + |\xi_1|^{p_1})\tilde{\gamma}_1(z, x_1)\Theta - \dot{\hat{\Theta}}\tilde{\Theta}.\end{aligned} \tag{3.28}$$

Similar to the argument in the proof of Theorem 3.4, one deduces from Lemma 2.4 that

$$\begin{aligned}&|\xi_1|(\|z\|^{p_0} + |\xi_1|^{p_0})\gamma_0(z, x_1)\Theta\\ &\quad \leq \left[\frac{\|z\|^{p_0+1}}{1+\hat{\Theta}^2} + \xi_1^{p_0+1}\omega_1(z, x_1)\right]\Theta\\ &\quad \leq \frac{\|z\|^{p_0+1}}{2} + \xi_1^{p_0+1}\omega_1(z, x_1)\sqrt{1+\hat{\Theta}^2}\\ &\qquad + \left[\frac{\|z\|^{p_0+1}}{1+\hat{\Theta}^2} + \xi_1^{p_0+1}\omega_1(z, x_1)\right]\tilde{\Theta}\end{aligned} \tag{3.29}$$

for a smooth function $\omega_1(z, x_1) \geq 0$. Likewise

$$\begin{aligned}&|\xi_1^{p_0-p_1+1}|(\|z\|^{p_1} + |\xi_1|^{p_1})\tilde{\gamma}_1(z, x_1)\Theta\\ &\quad \leq \frac{\|z\|^{p_0+1}}{2} + \xi_1^{p_0+1}\tilde{\omega}_1(z, x_1)\sqrt{1+\hat{\Theta}^2}\\ &\qquad + \left[\frac{\|z\|^{p_0+1}}{1+\hat{\Theta}^2} + \xi_1^{p_0+1}\tilde{\omega}_1(\cdot)\right]\tilde{\Theta}\end{aligned} \tag{3.30}$$

where $\tilde{\omega}_1(z, x_1) \geq 0$ is a smooth function.

Substituting (3.29) and (3.30) into (3.28) yields

$$\begin{aligned}\dot{V}_1 \leq &-r\|z\|^{p_0+1} + \xi_1^{p_0-p_1+1}x_2^{p_1}\\ &+ \xi_1^{p_0+1}(\omega_1(z, x_1) + \tilde{\omega}_1(z, x_1))\sqrt{1+\hat{\Theta}^2}\\ &+ (\eta_1(\cdot) + \tilde{\Theta})\left(\Psi_1(\cdot) - \dot{\hat{\Theta}}\right)\end{aligned}$$

where $\eta_1(\cdot) = 0$ and

$$\Psi_1(\cdot) = \frac{2\|z\|^{p_0+1}}{1+\hat{\Theta}^2} + \xi_1^{p_0+1}(\omega_1(z, x_1) + \tilde{\omega}_1(z, x_1)).$$

Observe that the C^∞ virtual controller

$$x_2^* = -\xi_1(r + (\omega_1(z, x_1) + \tilde{\omega}_1(z, x_1))\sqrt{1+\hat{\Theta}^2})^{1/p_1}$$

renders

$$\begin{aligned}\dot{V}_1 \leq &-r\left(\|z\|^{p_0+1} + \xi_1^{p_0+1}\right) + \xi_1^{p_0-p_1+1}(x_2^{p_1} - x_2^{*p_1})\\ &+ (\eta_1 + \tilde{\Theta})\left(\Psi_1(z, x_1) - \dot{\hat{\Theta}}\right)\end{aligned}$$

which completes the proof of Step 1.

The remaining part of the proof is analogous to that of Theorem 3.4 and is, therefore, omitted. ∎

In the next section, we shall prove that all the assumptions of Theorem 3.10 are automatically satisfied for partially feedback linearizable systems with a triangular structure, and hence they are nothing but a *high-order version of partial feedback linearizable condition*.

C. Feedback Linearizable Systems

So far, we have investigated adaptive control of high-order triangular systems with nonlinear parameterization. We now discuss a special case of (3.1), whose adaptive regulation with global stability is rather important and has occupied a central role in the nonlinear adaptive control literature.

Consider a class of C^1 nonlinearly parameterized, feedback linearizable systems of the form

$$\begin{aligned}\dot{x}_1 &= x_2 + \phi_1(x_1, \theta)\\ &\vdots\\ \dot{x}_{n-1} &= x_n + \phi_{n-1}(x_1, \ldots, x_{n-1}, \theta)\\ \dot{x}_n &= u + \phi_n(x_1, \ldots, x_n, \theta).\end{aligned} \tag{3.31}$$

In the literature of which we are aware, only few results studied the adaptive control problem of (3.31); see, for instance, [2], [23], and [4], in which adaptive control of a subclass of systems (3.31) was investigated, under the restrictive *convex/concave* parameterization condition. When $\phi_i(x_1, \ldots, x_i, \theta) = \phi_i(x_1, \ldots, x_i)\theta$, (3.31) reduces to a feedback linearizable system with linear parameterization for which adaptive regulation was addressed in [14], [24], [25], and [18].

In what follows, we illustrate that *without imposing any condition,* global adaptive stabilization of the nonlinearly parameterized system (3.31) is indeed possible. As a matter of fact, using Theorem 3.4 it is straightforward to deduce the following important conclusion which was recently proved in [22].

Corollary 3.11 [22]: For nonlinearly parameterized feedback linearizable systems (3.31), where $\phi_i(0, \theta) = 0$ for all $\theta \in R^s$, the problem of adaptive regulation with global stability is solvable by a C^∞ 1-D adaptive controller of the form (3.4).

Proof: The result is a direct consequence of Theorem 3.4. Obviously, Assumption 3.1 holds automatically for feedback linearizable systems (3.31) because $p_1 = \cdots = p_n = 1$. Since $\phi_i(x_1, \ldots, x_i, \theta)$ is C^1 and $\phi_i(0, \theta) = 0$, using the identity $F(X) - F(0) \equiv A(X)X$, with $X \in \mathbb{R}^m$ and $A(X) \in \mathbb{R}^{m\times m}$, yields

$$\phi_i(x_1, \ldots, x_i, \theta) = \sum_{j=1}^{i} x_j a_{i,j}(x_1, \ldots, x_i, \theta) \quad \text{for } C^0 \text{ functions } a_{i,j}(\cdot),\ i = 1, \ldots, n.$$

This, in turn, implies the existence of a continuous function $b_i(x_1, \ldots, x_i, \theta) \geq 0$, such that Assumtpion 3.2 is satisfied. Therefore, Corollary 3.11 follows immediately from Theorem 3.4. ∎

Remark 3.12: Due to the nature of the feedback domination design, it is not difficult to conclude that Corollary 3.11 remains true for the following uncertain C^0 system:

$$\dot{x}_i = x_{i+1} + \phi_i(x, t, \theta) \quad x_{n+1} := u, \qquad i = 1, \ldots, n \tag{3.32}$$

as long as there exist continuous functions $b_i(x_1, \ldots, x_i, \theta) \geq 0$, $i = 1, \ldots, n$, such that

$$|\phi_i(x, t, \theta)| \leq (|x_1| + \cdots + |x_i|)b_i(x_1, \ldots, x_i, \theta). \tag{3.33}$$

The following seemingly simple yet nontrivial example illustrates the application of Remark 3.12 and Corollary 3.11.

Example 3.13: Consider the nonlinearly parameterized system

$$\begin{aligned}\dot{x}_1 &= x_2 + \frac{\theta_1 x_1^2}{(1+\theta_2 x_3)^2 + x_3^2}\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\end{aligned} \tag{3.34}$$

where θ_1 and θ_2 are unknown parameters.

This system does not have a triangular structure but is of the form (3.32). Observe that $\phi_1(x, \theta) = \theta_1 x_1^2/((1+\theta_2 x_3)^2 + x_3^2)$ satisfies the condition (3.33). Indeed, a direct calculation gives

$$|\phi_1(x, \theta)| \leq |x_1| b_1(x_1, \theta) = x_1^2 \Theta$$

with

$$b_1(x_1, \theta) = |x_1|\Theta \quad \Theta = |\theta_1|(1+\theta_2^2). \tag{3.35}$$

By Remark 3.12, global adaptive regulation of the nonlinearly parameterized system (3.34) is solvable by a smooth adaptive controller. In what follows we illustrate how a smooth, 1-D adaptive controller (3.4) can be explicitly constructed for (3.34). We begin by considering $V_1(x_1, \tilde{\Theta}) = (x_1^2/2) + (\tilde{\Theta}^2/2)$, $\tilde{\Theta} = \Theta - \hat{\Theta}$. A straightforward computation shows that

$$\begin{aligned}\dot{V}_1 &\leq x_1 x_2 + x_1^2\sqrt{1+x_1^2}\,\Theta - \dot{\hat{\Theta}}\tilde{\Theta}\\ &\leq -3x_1^2 + x_1(x_2 - x_2^*) + \left(\Psi_1(x_1) - \dot{\hat{\Theta}}\right)\tilde{\Theta}\end{aligned}$$

where

$$x_2^*(x_1, \hat{\Theta}) = -x_1\left(3 + \sqrt{1+x_1^2}\,\hat{\Theta}\right)$$

$$\Psi_1(x_1) = x_1^2\sqrt{1+x_1^2}.$$

Next, choose $V_2(x_1, \xi_2, \tilde{\Theta}) = V_1(x_1, \tilde{\Theta}) + (\xi_2^2/2)$, with $\xi_2 = x_2 - x_2^*$. Then

$$\begin{aligned}\dot{V}_2 \leq{}& -3x_1^2 + \left(\Psi_1(x_1) - \dot{\hat{\Theta}}\right)\tilde{\Theta} + \xi_2\left(x_3 + x_1 - \frac{\partial x_2^*}{\partial x_1}x_2\right)\\ &- \xi_2\frac{\partial x_2^*}{\partial x_1}\phi_1(\cdot) - \xi_2\frac{\partial x_2^*}{\partial \hat{\Theta}}\dot{\hat{\Theta}}.\end{aligned} \tag{3.36}$$

By (3.35)

$$\begin{aligned}\left|\xi_2\frac{\partial x_2^*}{\partial x_1}\phi_1(\cdot)\right| &\leq |\xi_2|\left|\frac{\partial x_2^*}{\partial x_1}\right| x_1^2\Theta\\ &\leq x_1^2 + \xi_2^2 x_1^2\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\frac{\sqrt{1+\hat{\Theta}^2}}{4}\hat{\Theta}\\ &\quad + A_2(x_1, x_2, \hat{\Theta})\tilde{\Theta}\end{aligned} \tag{3.37}$$

where

$$A_2(x_1, x_2, \hat{\Theta}) = \frac{x_1^2}{\sqrt{1+\hat{\Theta}^2}} + \xi_2^2 x_1^2 \left(\frac{\partial x_2^*}{\partial x_1}\right)^2 \frac{\sqrt{1+\hat{\Theta}^2}}{4}.$$

Substituting (3.37) into (3.36) yields

$$\dot{V}_2 \le -2x_1^2 - \xi_2^2 + \left(\Psi_2(x_1, x_2, \hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}\right) + \xi_2(x_3 - x_3^*) \quad (3.38)$$

where

$$\Psi_2(x_1, x_2, \hat{\Theta}) = \Psi_1(x_1) + A_2(x_1, x_2, \hat{\Theta})$$
$$x_3^* = -\xi_2 - x_1 + \frac{\partial x_2^*}{\partial x_1} x_2 - \xi_2 x_1^2 \left(\frac{\partial x_2^*}{\partial x_1}\right)^2 \cdot \frac{\sqrt{1+\hat{\Theta}^2}}{4}\hat{\Theta} + \frac{\partial x_2^*}{\partial \hat{\Theta}} \Psi_2(x_1, x_2, \hat{\Theta}).$$

At the last step, consider $V_3(x_1, \xi_2, \xi_3, \tilde{\Theta}) = V_2(x_1, \xi_2, \tilde{\Theta}) + (\xi_3^2/2)$, $\xi_3 = x_3 - x_3^*$. Clearly

$$\dot{V}_3 = \dot{V}_2 + \xi_3 \left(u - \frac{\partial x_3^*}{\partial x_1}(x_2 + \phi_1(\cdot)) - \frac{\partial x_3^*}{\partial x_2} x_3 - \frac{\partial x_3^*}{\partial \hat{\Theta}} \dot{\hat{\Theta}}\right). \quad (3.39)$$

Similar to the estimate (3.37), we have

$$\left|\xi_3 \frac{\partial x_3^*}{\partial x_1}\phi_1(\cdot)\right| \le \frac{x_1^2}{2} + \frac{1}{2}\xi_3^2 x_1^2 \left(\frac{\partial x_3^*}{\partial x_1}\right)^2 \cdot \sqrt{1+\hat{\Theta}^2 + \left(\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2}\hat{\Theta} + A_3(x_1, x_2, x_3, \hat{\Theta})\tilde{\Theta} \quad (3.40)$$

where

$$A_3(x_1, x_2, x_3, \hat{\Theta}) = \frac{x_1^2}{2}\left[\left(\sqrt{1+\hat{\Theta}^2 + \left(\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2}\right)^{-1} + \xi_3^2 \left(\frac{\partial x_3^*}{\partial x_1}\right)^2 \sqrt{1+\hat{\Theta}^2 + \left(\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2}\right].$$

This, together with (3.38) and (3.39), implies

$$\dot{V}_3 \le -\tfrac{3}{2}x_1^2 - \xi_2^2 + \left(\Psi_2(x_1, x_2, \hat{\Theta}) + A_3(\cdot) - \dot{\hat{\Theta}}\right) \cdot \left(\tilde{\Theta} + \xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}} + \xi_3 \frac{\partial x_3^*}{\partial \hat{\Theta}}\right) + \xi_3 v - \xi_3 \frac{\partial x_3^*}{\partial \hat{\Theta}}(\Psi_2(\cdot) + A_3(\cdot)) - A_3(\cdot)\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}$$

where v is the new control input satisfying

$$u = v - \left(\xi_2 - \frac{\partial x_3^*}{\partial x_1}x_2 - \frac{\partial x_3^*}{\partial x_2}x_3 + \frac{1}{2}\xi_3 x_1^2 \left(\frac{\partial x_3^*}{\partial x_1}\right)^2 \hat{\Theta} \cdot \sqrt{1+\hat{\Theta}^2 + \left(\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2}\right). \quad (3.41)$$

Note that

$$-A_3(\cdot)\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}} \le \frac{x_1^2}{2} - \frac{1}{2}\xi_3^2 x_1^2 \left(\frac{\partial x_3^*}{\partial x_1}\right)^2 \cdot \sqrt{1+\hat{\Theta}^2 + \left(\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2}\,\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}.$$

Hence

$$\dot{V}_3 \le -x_1^2 - \xi_2^2 + \left(\Psi_2(\cdot) + A_3(\cdot) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}} + \xi_3 \frac{\partial x_3^*}{\partial \hat{\Theta}}\right) + \xi_3 \left(v - \frac{\partial x_3^*}{\partial \hat{\Theta}}(\Psi_2(\cdot) + A_3(\cdot)) - \frac{1}{2}\xi_3 x_1^2 \left(\frac{\partial x_3^*}{\partial x_1}\right)^2 \cdot \sqrt{1+\hat{\Theta}^2 + \left(\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2}\,\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}\right).$$

Clearly, the smooth adaptive controller

$$\dot{\hat{\Theta}} = \Psi_2(x_1, x_2, \hat{\Theta}) + A_3(x_1, x_2, x_3, \hat{\Theta})$$
$$v = -\xi_3 + \frac{\partial x_3^*}{\partial \hat{\Theta}}(\Psi_2(\cdot) + A_3(\cdot)) + \frac{1}{2}\xi_3 x_1^2 \left(\frac{\partial x_3^*}{\partial x_1}\right)^2 \sqrt{1+\hat{\Theta}^2 + \left(\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2}\,\xi_2 \frac{\partial x_2^*}{\partial \hat{\Theta}} \quad (3.42)$$

is such that $\dot{V}_3(x_1, \xi_2, \xi_3, \tilde{\Theta}) \le -x_1^2 - \xi_2^2 - \xi_3^2$. Hence, the 1-D C^∞ adaptive controller (3.41) and (3.42) makes the nonlinearly parameterized system (3.34) globally stable, with asymptotic state regulation.

We conclude this section by extending Corollary 3.11 to a class of cascade systems

$$\begin{aligned}\dot{z} &= f_0(z, x_1) + x_1 f_1(z, x_1, \theta)\\ \dot{x}_1 &= x_2 + \phi_1(z, x_1, \theta)\\ &\vdots\\ \dot{x}_r &= u + \phi_r(z, x_1, \ldots, x_n, \theta)\end{aligned} \quad (3.43)$$

where $z \in \mathbb{R}^{n-r}$, $\phi_i(\cdot)$, $i = 1, \ldots, r$, and $f_k(\cdot)$, $k = 0, 1$, are C^1 functions with $\phi_i(0, \ldots, 0, \theta) = 0 \; \forall \theta \in R^s$ and $f_0(0, 0) = 0$.

Under mild conditions on the zero dynamics of (3.43), it is possible to prove that global adaptive regulation is achievable for the cascade system (3.43). Indeed, the following result can be deduced from Theorem 3.10.

Theorem 3.14: Consider a nonlinearly parameterized cascade system (3.43). If there exists a C^2 Lyapunov function $U(z)$, which is positive–definite and proper, such that

$$\frac{\partial U}{\partial z} f(z, 0) \le -\|z\|^2. \quad (3.44)$$

Then, the problem of adaptive regulation with global stability is solvable by a smooth, 1-D adaptive controller of the form (3.4).

Proof: The proof is carried out by simply verifying that all the hypotheses of Theorem 3.10 are satisfied in the case of cascade systems (3.43). First of all, *A3.8* is clearly true because $p_1 = \cdots = p_r = 1$. By hypothesis, the C^1 functions $\phi_i(z, x_1, \ldots, x_i, \theta)$, $i = 1, \ldots, r$, can be decomposed as

$$\phi_i(z, x_1, \ldots, x_i, \theta) = z^T a_{i,0}(z, x_1, \ldots, x_i, \theta) + \sum_{k=1}^{i} x_k a_{i,k}(z, x_1, \ldots, x_i, \theta) \quad \text{for } C^0 \text{ functions } a_{i,k}(\cdot)$$

which leads to (3.26) with $p_i = 1$, $i = 1, \ldots, r$. Due to the same reasoning, there exist continuous functions $g_0(z, x_1)$ and $g_1(z, x_1)$ such that

$$f(z, x_1, \theta) := f_0(z, x_1) + x_1 f_1(z, x_1, \theta) \equiv g_0(z, x_1)z + x_1 g_1(z, x_1) + x_1 f_1(z, x_1, \theta)$$

which implies (3.25) with $p_1 = 1$. In other words, system (3.43) satisfies *A3.9* as well. Finally, it is easy to see from (3.44) that the condition (3.27) in Theorem 3.10 holds for the choice

$$p_0 = 1 \quad \alpha(z) = 0$$
$$\varphi(z, x_1, \theta) = \left\|\frac{\partial U}{\partial z}\right\| \left(\left\|\frac{f_0(z, x_1) - f_0(z, 0)}{x_1} + f_1(z, x_1, \theta)\right\|\right).$$

Clearly, $\varphi(z, x_1, \theta)$ is continuous. Since $(\partial U/\partial z)(0) = 0$, it follows from the Taylor expansion formula that there is a C^0 function $h(z, x_1, \theta) \geq 0$, such that

$$0 \leq \varphi(z, x_1, \theta) \leq \|z\| h(z, x_1, \theta) \leq (\|z\| + |x_1|) h(z, x_1, \theta).$$

According to the previous discussions, we conclude that all the conditions of Theorem 3.10 hold when $p_0 = p_1 = \cdots = p_r = 1$. Hence, Theorem 3.14 follows from Theorem 3.10. ∎

Remark 3.15: In the recent work [22], global adaptive stabilization has been shown to be possible for a larger class of cascade systems with nonlinear parameterization than system (3.43). Note that Theorem 3.14 remains true if (3.44) is replaced by the condition that $\dot{z} = f_0(z, 0)$ is GAS and LES.

IV. Nonlinearly Parameterized Systems Beyond a Triangular Structure

The main focus so far has been on the problem of adaptive regulation with global stability for a class of triangular systems with nonlinear parameterization. We now turn our attention to investigating the possibility of extending the adaptive control results obtained in the previous section to a larger class of nonlinearly parameterized systems such as (2.4), which go *beyond a lower triangular* form.

To design a globally stabilizing adaptive controller for systems (2.4), we need introducing a set of sufficient conditions that characterize a subclass of nonlinearly parameterized systems (2.4).

Assumption 4.1: There exist C^∞ functions $\lambda_i(x_1, \ldots, x_i) > 0$ and $\mu_i(x_1, \ldots, x_{i+1}, \theta)$, such that

$$\lambda_i(x_1, \ldots, x_i) \leq d_i(x, u, \theta) \leq \mu_i(x_1, \ldots, x_{i+1}, \theta), \quad i = 1, \ldots, n. \tag{4.1}$$

Assumption 4.2: For $i = 1, \ldots, n$, there exist $\varphi_{i,j}(x_1, \ldots, x_i, \theta)$ such that

$$\phi_i(x_1, \ldots, x_{i+1}, \theta) = \sum_{j=0}^{p_i - 1} x_{i+1}^j \varphi_{i,j}(x_1, \ldots, x_i, \theta) \tag{4.2}$$

$$|\varphi_{i,j}(x_1, \ldots, x_i, \theta)| \leq (|x_1|^{p_i - j} + \cdots + |x_i|^{p_i - j}) \cdot b_{i,j}(x_1, \ldots, x_i, \theta). \tag{4.3}$$

where $b_{i,j}(x_1, \ldots, x_i, \theta) \geq 0$, $j = 0, 1, \ldots, p_i - 1$, are continuous functions.

The main result of this section is the following theorem which generalizes Theorem 3.4.

Theorem 4.3: Under Assumptions 3.1, 4.1, and 4.2, there is a smooth, *1-D* adaptive controller of the form (3.4), which solves the problem of adaptive regulation with global stability for nonlinearly parameterized systems (2.4).

Before proving Theorem 4.3, we first introduce a very useful lemma.

Lemma 4.4: For the uncertain nonlinear functions $d_i(\cdot)$ and $\phi_i(\cdot)$ satisfying Assumptions 4.1 and 4.2, respectively, there are a constant $\Theta \geq 1$ and C^∞ functions $\tilde{\gamma}_i(x_1, \ldots, x_{i+1}) \geq 0$, $\gamma_i(x_1, \ldots, x_i) \geq 0$, such that

$$d_i(x, u, \theta) \leq \tilde{\gamma}_i(x_1, \ldots, x_{i+1})\Theta \tag{4.4}$$

$$|\phi_i(x_1, \ldots, x_{i+1}, \theta)| \leq \frac{\lambda_i(x_1, \ldots, x_i)}{2} |x_{i+1}|^{p_i} + (|x_1|^{p_i} + \cdots + |x_i|^{p_i}) \cdot \gamma_i(x_1, \ldots, x_i)\Theta. \tag{4.5}$$

Proof: When $p_i > 1$, for $k = 1, \ldots, i$, using Lemma 2.4 yields

$$\left|x_{i+1}^j x_k^{p_i - j} b_{i,j}(\cdot)\right| \leq \frac{j}{p_i} |x_{i+1}|^{p_i} \pi(\cdot) + \frac{p_i - j}{p_i} |x_k|^{p_i} \cdot \left[b_{i,j}^{p_i}(\cdot)\pi^{-j}(\cdot)\right]^{1/(p_i - j)}$$

where $\pi(\cdot) > 0$ is a smooth function to be determined later. Thus

$$\left|x_{i+1}^j \varphi_{i,j}(\cdot)\right| \leq \frac{ij}{p_i} |x_{i+1}|^{p_i} \pi(\cdot) + \frac{p_i - j}{p_i} \cdot \left[b_{i,j}^{p_i}(\cdot)\pi^{-j}(\cdot)\right]^{1/(p_i - j)} \sum_{k=1}^{i} |x_k|^{p_i}. \tag{4.6}$$

Combining (4.6) with (4.2), we have

$$|\phi_i(x_1, \ldots, x_{i+1}, \theta)| \leq \frac{i}{p_i} |x_{i+1}|^{p_i} \pi(\cdot) \sum_{j=0}^{p_i - 1} j + (|x_1|^{p_i} + \cdots + |x_i|^{p_i}) \sum_{j=0}^{p_i - 1} \cdot \frac{p_i - j}{p_i} \left[b_{i,j}^{p_i}(\cdot)\pi^{-j}(\cdot)\right]^{1/(p_i - j)}. \tag{4.7}$$

Choose

$$\pi(x_1, \ldots, x_i) = \frac{p_i \lambda_i(x_1, \ldots, x_i)}{2i \sum_{j=1}^{p_i - 1} j} > 0.$$

Then

$$\begin{aligned}|\phi_i(x_1, \ldots, x_{i+1}, \theta)| &\\ \le \frac{\lambda_i(\cdot)}{2}|x_{i+1}|^{p_i} + (|x_1|^{p_i} + \cdots + |x_i|^{p_i}) &\\ \cdot B_i(x_1, \ldots, x_i, \theta), \qquad \text{for a } C^0\ B_i(\cdot). &\quad (4.8)\end{aligned}$$

Clearly, the previous inequality also holds when $p_i = 1$ [i.e., by choosing $B_i(\cdot) = b_{i,0}(\cdot)$].

Now, it is deduced from (4.8) and Lemma 2.1 that there are smooth functions $\gamma_i(x_1, \ldots, x_i)$, $\tilde{\gamma}_i(x_1, \ldots, x_{i+1})$, $D_i(\theta)$ and $\tilde{D}_i(\theta)$ such that

$$\begin{aligned}B_i(x_1, \ldots, x_i, \theta) &\le \gamma_i(x_1, \ldots, x_i)D_i(\theta)\\ d_i(x, u, \theta) &\le \mu_i(x_1, \ldots, x_{i+1}, \theta)\\ &\le \tilde{\gamma}_i(x_1, \ldots, x_{i+1})\tilde{D}_i(\theta).\end{aligned}$$

Set $\Theta = \sum_{i=1}^{n}(D_i(\theta) + \tilde{D}_i(\theta))$. Then, (4.4) and (4.5) follow immediately. ∎

Proof of Theorem 4.3: The proof is based on a combination of *adding a power integrator*, Lemma 2.1 and Lemma 4.4, in the spirit of Theorem 3.4.

Initial Step: Let $\tilde{\Theta}(t) = \Theta - \hat{\Theta}(t)$, where $\Theta \ge 1$ be the unknown constant defined in Lemma 4.4. Consider $V_1(x_1, \tilde{\Theta}) = (1/2)x_1^2 + (1/2)\tilde{\Theta}^2$. By Lemma 4.4, it is clear that x_1—subsystem of (2.4) satisfies

$$\begin{aligned}\dot{V}_1(x_1, \tilde{\Theta}) \le d_1(x, u, \theta)x_1x_2^{p_1} + \frac{\lambda_1(\cdot)}{2}|x_1x_2^{p_1}|&\\ + x_1^{p_1+1}\gamma_1(x_1)(\tilde{\Theta} + \hat{\Theta}) - \dot{\hat{\Theta}}(t)\tilde{\Theta}(t).&\end{aligned}$$

With the choice of the smooth virtual controller

$$\begin{aligned}x_2^* &= -x_1\left[\frac{2n + 2\gamma_1(x_1)\sqrt{\hat{\Theta}^2+1}}{\lambda_1(x_1)}\right]^{1/p_1}\\ &:= -x_1\beta_1(x_1, \hat{\Theta})\end{aligned}$$

we have

$$\begin{aligned}\dot{V}_1(x_1, \tilde{\Theta}) &\le -nx_1^{p_1+1} + d_1(x, u, \theta)x_1x_2^{p_1}\\ &+ \frac{\lambda_1(\cdot)}{2}|x_1x_2^{p_1}| - \frac{\lambda_1(\cdot)}{2}x_1x_2^{*p_1} + \left(\Psi_1(x_1) - \dot{\hat{\Theta}}(t)\right)\tilde{\Theta}(t)\end{aligned}$$

where $\Psi_1(x_1) = x_1^{p_1+1}\gamma_1(x_1) \ge 0$. Since $x_1x_2^{*p_1} \le 0$, it is easy to deduce from Assumption 4.1 that

$$\begin{aligned}\dot{V}_1(x_1, \tilde{\Theta}) \le& -nx_1^{p_1+1} + d_1(\cdot)x_1x_2^{p_1} - \lambda_1(\cdot)x_1x_2^{*p_1}\\ &+ \frac{\lambda_1(\cdot)}{2}|x_1x_2^{p_1}| - \frac{\lambda_1(\cdot)}{2}|x_1x_2^{*p_1}|\\ &+ \left(\Psi_1(x_1) - \dot{\hat{\Theta}}(t)\right)\tilde{\Theta}(t)\\ \le& -nx_1^{p_1+1} + \left(\mu_1(x_1, x_2, \theta) + \frac{\lambda_1(x_1)}{2}\right)|x_1|\\ &\cdot|x_2^{p_1} - x_2^{*p_1}| + \left(\Psi_1(x_1) - \dot{\hat{\Theta}}(t)\right)(\tilde{\Theta}(t) + \eta_1)\end{aligned} \quad (4.9)$$

with $\eta_1 = 0$.

Inductive Step: Suppose for system (2.4) with dimension k, there are a set of smooth virtual controllers $x_1^*, \ldots, x_{k+1}^*$, defined by (3.6), such that

$$\begin{aligned}&\dot{V}_k(\xi_1, \ldots, \xi_k, \tilde{\Theta})\Big|_{(2.4)}\\ &\le -(n-k+1)\left(\xi_1^{p_1+1} + \cdots + \xi_k^{p_1+1}\right)\\ &+ \left(\mu_k(\cdot) + \frac{\lambda_k(\cdot)}{2}\right)\left|\xi_k^{p_1-p_k+1}\right|\,|x_{k+1}^{p_k} - x_{k+1}^{*p_k}|\\ &+ \left(\Psi_k(\xi_1, \ldots, \xi_k, \hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\xi_1, \ldots, \xi_k, \hat{\Theta})\right)\end{aligned} \quad (4.10)$$

where

$$V_k(\xi_1, \ldots, \xi_k, \tilde{\Theta}) := \sum_{j=1}^{k}\frac{\xi_j^{p_1-p_j+2}}{p_1 - p_j + 2} + \frac{\tilde{\Theta}^2}{2}$$

is a positive–definite and proper Lyapunov function. Moreover

$$\begin{aligned}0 &\le \Psi_k(\xi_1, \ldots, \xi_k, \hat{\Theta})\\ &\le \left(\xi_1^{p_1+1} + \cdots + \xi_k^{p_1+1}\right)\alpha_k(\xi_1, \ldots, \xi_k, \hat{\Theta})\\ &\quad \text{for } C^\infty \alpha_k(\cdot) \ge 0.\end{aligned} \quad (4.11)$$

Then, (4.10) and (4.11) are also true when the dimension of system (2.4) is equal to $k+1$. To prove this claim, consider the Lyapunov function

$$V_{k+1}(\xi_1, \ldots, \xi_{k+1}, \tilde{\Theta}) = V_k(\xi_1, \ldots, \xi_k, \tilde{\Theta}) + \frac{\xi_{k+1}^{p_1-p_{k+1}+2}}{p_1 - p_{k+1} + 2}.$$

Clearly, taking the time derivative of V_{k+1} along the solutions of the $k+1$-dimensional system (2.4) gives

$$\begin{aligned}&\dot{V}_{k+1}(\cdot)\\ &\le -(n-k+1)\left(\xi_1^{p_1+1} + \cdots + \xi_k^{p_1+1}\right)\\ &+ \left(\mu_k(\cdot) + \frac{\lambda_k(\cdot)}{2}\right)\left|\xi_k^{p_1-p_k+1}\right|\,|x_{k+1}^{p_k} - x_{k+1}^{*p_k}|\\ &+ \left(\Psi_k(\cdot) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\cdot)\right) + \xi_{k+1}^{p_1-p_{k+1}+1}\\ &\cdot\left[d_{k+1}(\cdot)x_{k+2}^{p_{k+1}} + \phi_{k+1}(\cdot) - \sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_j}\dot{x}_j - \frac{\partial x_{k+1}^*}{\partial \hat{\Theta}}\dot{\hat{\Theta}}\right].\end{aligned} \quad (4.12)$$

Combining the estimation method in Theorem 3.4 with Lemma 4.4, one can prove that there is a smooth function $\overline{\omega}_{k+1}(\cdot) \ge 0$, such that

$$\begin{aligned}&\left|\xi_{k+1}^{p_1-p_{k+1}+1}\right|\left|\phi_{k+1}(\cdot) - \sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_j}\dot{x}_j\right|\\ &\le \frac{\lambda_{k+1}(\cdot)}{2}\left|\xi_{k+1}^{p_1-p_{k+1}+1}x_{k+2}^{p_{k+1}}\right| + \frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{6}\\ &+ \xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\cdot)\sqrt{\hat{\Theta}^2+1}\end{aligned}$$

$$+\left[\frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} +\xi_{k+1}^{p_1+1}\overline{\omega}_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta})\right]\tilde{\Theta}. \qquad (4.13)$$

Similarly, the following estimates hold for a smooth function $\omega_{k+1}(\cdot)\ge 0$:

$$\begin{aligned}
&\left(\mu_k(\cdot)+\frac{\lambda_k(\cdot)}{2}\right)\left|\xi_k^{p_1-p_k+1}\right|\left|x_{k+1}^{p_k}-x_{k+1}^{*p_k}\right| \\
&\le\left(\tilde{\gamma}_k(x_1,\,\ldots,\,x_{k+1})+\frac{\lambda_k(x_1,\,\ldots,\,x_k)}{2}\right) \\
&\quad\cdot\left|\xi_k^{p_1-p_k+1}\right|\left|x_{k+1}^{p_k}-x_{k+1}^{*p_k}\right|\Theta \\
&\le\frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{6}+\xi_{k+1}^{p_1+1}\omega_{k+1}(\cdot)\sqrt{\hat{\Theta}^2+1} \\
&\quad+\left[\frac{\sum_{l=1}^{k}\xi_l^{p_1+1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} +\xi_{k+1}^{p_1+1}\omega_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta})\right]\tilde{\Theta}. \qquad (4.14)
\end{aligned}$$

Substituting (4.13) and (4.14) into (4.12) yields

$$\begin{aligned}
\dot{V}_{k+1}(\cdot)\le&-\left(n-k+\tfrac{2}{3}\right)\left(\xi_1^{p_1+1}+\cdots+\xi_k^{p_1+1}\right) \\
&+d_{k+1}(\cdot)\xi_{k+1}^{p_1-p_{k+1}+1}x_{k+2}^{p_{k+1}} \\
&+\frac{\lambda_{k+1}(\cdot)}{2}\left|\xi_{k+1}^{p_1-p_{k+1}+1}x_{k+2}^{p_{k+1}}\right| \\
&+\xi_{k+1}^{p_1+1}[\overline{\omega}_{k+1}(\cdot)+\omega_{k+1}(\cdot)]\sqrt{\hat{\Theta}^2+1} \\
&+\left[\frac{2\sum_{l=1}^{k}\xi_l^{p_1+1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} +\xi_{k+1}^{p_1+1}(\overline{\omega}_{k+1}(\cdot)+\omega_{k+1}(\cdot))\right]\tilde{\Theta} \\
&-\xi_{k+1}^{p_1-p_{k+1}+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\dot{\hat{\Theta}} \\
&+\left(\Psi_k(\xi_1,\,\ldots,\,\xi_k,\,\hat{\Theta})-\dot{\hat{\Theta}}\right) \\
&\cdot\left(\tilde{\Theta}+\eta_k(\xi_1,\,\ldots,\,\xi_k,\,\hat{\Theta})\right). \qquad (4.15)
\end{aligned}$$

Define

$$\begin{aligned}
&\Psi_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta}) \\
&\quad=\Psi_k(\xi_1,\,\ldots,\,\xi_k,\,\hat{\Theta})+\frac{2\sum_{l=1}^{k}\xi_l^{p_1+1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} \\
&\qquad+\xi_{k+1}^{p_1+1}[\overline{\omega}_{k+1}(\cdot)+\omega_{k+1}(\cdot)] \\
&\eta_{k+1}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta}) \\
&\quad=\eta_k(\xi_1,\,\ldots,\,\xi_k,\,\hat{\Theta})+\xi_{k+1}^{p_1-p_{k+1}+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}.
\end{aligned}$$

Then, (4.15) can be rewritten as follows:

$$\begin{aligned}
&\dot{V}_{k+1}(\cdot) \\
&\quad\le-\left(n-k+\tfrac{2}{3}\right)\left(\xi_1^{p_1+1}+\cdots+\xi_k^{p_1+1}\right) \\
&\qquad+d_{k+1}(\cdot)\xi_{k+1}^{p_1-p_{k+1}+1}x_{k+2}^{p_{k+1}}+\frac{\lambda_{k+1}(\cdot)}{2}\left|\xi_{k+1}^{p_1-p_{k+1}+1}x_{k+2}^{p_{k+1}}\right| \\
&\qquad+\xi_{k+1}^{p_1+1}[\overline{\omega}_{k+1}(\cdot)+\omega_{k+1}(\cdot)]\sqrt{\hat{\Theta}^2+1} \\
&\qquad+\left(\Psi_{k+1}(\cdot)-\dot{\hat{\Theta}}\right)\left(\tilde{\Theta}+\eta_{k+1}(\cdot)\right) \\
&\qquad-\left[\frac{2\sum_{l=1}^{k}\xi_l^{p_1+1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))} +\xi_{k+1}^{p_1+1}(\overline{\omega}_{k+1}(\cdot)+\omega_{k+1}(\cdot))\right]\eta_k(\cdot) \\
&\qquad-\xi_{k+1}^{p_1-p_{k+1}+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\Psi_{k+1}(\cdot). \qquad (4.16)
\end{aligned}$$

Finally, it is not difficult to show that

$$\begin{aligned}
&\left|\left(\frac{2\sum_{l=1}^{k}\xi_l^{p_1+1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\cdot))}+\xi_{k+1}^{p_1+1}(\overline{\omega}_{k+1}(\cdot)+\omega_{k+1}(\cdot))\right)\eta_k(\cdot)\right. \\
&\qquad\left.+\xi_{k+1}^{p_1-p_{k+1}+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\Psi_{k+1}(\cdot)\right| \\
&\quad\le\frac{2}{3}\sum_{l=1}^{k}\xi_l^{p_1+1}+\xi_{k+1}^{p_1+1}(\overline{\omega}_{k+1}(\cdot)+\omega_{k+1}(\cdot))\sqrt{\eta_k^2(\cdot)+1} \\
&\qquad+\xi_{k+1}^{p_1+1}\sum_{j=1}^{k+1}\tilde{\beta}_{(k+1)j}(\xi_1,\,\ldots,\,\xi_{k+1},\,\hat{\Theta}). \qquad (4.17)
\end{aligned}$$

Substituting (4.17) into (4.16), we arrive at

$$\dot{V}_{k+1}(\cdot)\le-(n-k)\left(\xi_1^{p_1+1}+\cdots+\xi_k^{p_1+1}\right)$$

$$
\begin{aligned}
&+ d_{k+1}(\cdot)\xi_{k+1}^{p_1-p_{k+1}+1}x_{k+2}^{p_{k+1}} \\
&+ \frac{\lambda_{k+1}(\cdot)}{2}\left|\xi_{k+1}^{p_1-p_{k+1}+1}x_{k+2}^{p_{k+1}}\right| \\
&+ \xi_{k+1}^{p_1+1}\left((\overline{\omega}_{k+1}(\cdot)+\omega_{k+1}(\cdot))\sqrt{\hat{\Theta}^2+1}+\hat{\gamma}_{k+1}(\cdot)\right) \\
&+ \left(\Psi_{k+1}(\cdot)-\dot{\hat{\Theta}}\right)\left(\tilde{\Theta}+\eta_{k+1}(\cdot)\right) \qquad (4.18)
\end{aligned}
$$

where $\hat{\gamma}_{k+1}(\cdot) = (\overline{\omega}_{k+1}(\cdot) + \omega_{k+1}(\cdot)) \sqrt{\eta_k^2(\cdot)+1} + \sum_{j=1}^{k+1} \tilde{\beta}_{(k+1)j}(\xi_1, \ldots, \xi_{k+1}, \hat{\Theta}) \geq 0$ is a smooth function.

Now, it is easy to see the smooth virtual controller

$$
x_{k+2}^* = -\xi_{k+1}\left[\frac{2n-2k+2\rho_{k+1}(\xi_1, \ldots, \xi_{k+1}, \hat{\Theta})}{\lambda_{k+1}(x_1, \ldots, x_{k+1})}\right]^{1/p_{k+1}} \qquad (4.19)
$$

with $\rho_{k+1}(\cdot) := (\overline{\omega}_{k+1}(\cdot)+\omega_{k+1}(\cdot)) \sqrt{\hat{\Theta}^2+1}+\hat{\gamma}_{k+1}(\cdot) \geq 0$ being smooth, renders

$$
\begin{aligned}
&\dot{V}_{k+1}(\xi_1, \ldots, \xi_{k+1}, \tilde{\Theta}) \\
&\quad \leq -(n-k)\left(\xi_1^{p_1+1}+\cdots+\xi_{k+1}^{p_1+1}\right) \\
&\qquad + \left(\mu_{k+1}(\cdot)+\frac{\lambda_{k+1}(\cdot)}{2}\right)\left|\xi_{k+1}^{p_1-p_{k+1}+1}\right| \left|x_{k+2}^{p_{k+1}}-x_{k+2}^{*p_{k+1}}\right| \\
&\qquad + \left(\Psi_{k+1}(\xi_1, \ldots, \xi_{k+1}, \hat{\Theta})-\dot{\hat{\Theta}}\right) \\
&\qquad \cdot \left(\tilde{\Theta}+\eta_{k+1}(\xi_1, \ldots, \xi_{k+1}, \hat{\Theta})\right). \qquad (4.20)
\end{aligned}
$$

This completes the proof of the inductive step, from which a smooth, 1-D adaptive controller can be easily constructed for the nonlinearly parameterized system (2.4). ∎

From a combination of Theorem 4.3 and Lemma 4.4, it is not difficult to deduce the following result.

Corollary 4.5: Under Assumptions 3.1 and 4.1, and (4.5), the problem of global adaptive regulation for nonlinearly parameterized systems (2.4) is solvable by the smooth adaptive controller (3.4). ∎

When $p_i = 1$, $d_i(\cdot) = 1$, $i = 1, \ldots, n$, (2.4) reduces to the nonlinearly parameterized system with controllable linearization

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \phi_1(x_1, x_2, \theta) \\
\dot{x}_2 &= x_3 + \phi_2(x_1, x_2, x_3, \theta) \\
&\vdots \\
\dot{x}_n &= u + \phi_n(x_1, \ldots, x_n, u, \theta). \qquad (4.21)
\end{aligned}
$$

It is worthwhile pointing out that the problem of adaptive regulation with global stability remains unsolved even in the case where the unknown parameter θ appears linearly in (4.21). For the linearly parameterized system (4.21), only *local* adaptive regulation results were obtained [14], [25]. However, using the new design technique proposed in Theorem 4.3 or Corollary 4.5, one is able to derive a sufficient condition under which a globally stabilizing adaptive controller can be explicitly constructed.

Corollary 4.6: The global adaptive regulation problem of (4.21) is solvable by a smooth, 1-D adaptive controller (3.4) if

$$
\begin{aligned}
&|\phi_i(x_1, \ldots, x_{i+1}, \theta)| \\
&\quad \leq a_i|x_{i+1}| + (|x_1| + \cdots + |x_i|)b_i(x_1, \ldots, x_i, \theta) \qquad (4.22)
\end{aligned}
$$

where $0 \leq a_i < 1$, for $i = 1, \ldots, n$, and $x_{n+1} := u$.

Proof: It follows immediately from Theorem 4.3 or Corollary 4.5. ∎

A nice application of Corollary 4.6 can be demonstrated by solving the adaptive regulation problem for a nonlinearly parameterized system with a nontriangular structure.

Example 4.7: Consider the planar system with nonlinear parameterization

$$
\begin{aligned}
\dot{x}_1 &= x_2 + x_1(1+x_2^2)^{1/3}\theta_1^{x_1}, \qquad \theta_1 > 0 \\
\dot{x}_2 &= u + \ln(1+(\theta_2 x_2)^2), \qquad \theta_2 \in \mathbb{R}. \qquad (4.23)
\end{aligned}
$$

The aforementioned system is of the form (4.21). Using Young's inequality, it is easy to prove that

$$
\begin{aligned}
|\phi_1(x, \theta)| &\leq |x_1| \left(1+x_2^{2/3}\right)|\theta_1^{x_1}| \\
&\leq |x_1|\,|\theta_1^{x_1}| + \tfrac{2}{3}\,|x_2| + \tfrac{1}{3}\,|x_1^3\theta_1^{3x_1}| \\
&\leq \tfrac{2}{3}\,|x_2| + |x_1|\,\left(1+\tfrac{1}{3}\,x_1^2\right)e^{(x_1^2/4)}e^{9\ln^2\theta_1}. \qquad (4.24)
\end{aligned}
$$

On the other hand, by the mean value theorem we have

$$
|\phi_2(x, \theta)| \leq |\theta_2|\,|x_2|. \qquad (4.25)
$$

Therefore, the condition (4.22) is fulfilled. By Corollary 4.6, global adaptive regulation of system (4.23) is solvable by the *1-D* smooth adaptive controller (3.4). A globally stabilizing smooth adaptive controller can be explicitly constructed, as briefly illustrated as follows.

Using (4.24) and (4.25), we define $\Theta = |\theta_2| + e^{9\ln^2\theta_1}$. Let $\hat{\Theta}$ be the estimate of Θ and consider the Lyapunov function

$$
V_2(x_1, x_2, \tilde{\Theta}) = \tfrac{1}{2}\left[x_1^2+(x_2-x_2^*)^2+(\hat{\Theta}-\Theta)^2\right]
$$

where $x_2^* = -x_1\beta_1(x_1, \hat{\Theta})$ and $\beta_1(x_1, \hat{\Theta}) = 6 + (3+x_1^2)e^{x_1^2/4}\sqrt{1+\hat{\Theta}^2}$.

Following the design procedure of Theorem 4.3, it can be shown that the smooth adaptive controller

$$
\begin{aligned}
\dot{\hat{\Theta}} &= x_1^2\left(1+\frac{1}{3}\,x_1^2\right)e^{x_1^2/4} + \frac{x_1^2}{2+2\hat{\Theta}^2} + (x_2-x_2^*)^2\rho_2(\cdot) \\
u &= -(x_2-x_2^*)\left(\tfrac{34}{9}+\rho_1(\cdot)+\rho_2(\cdot)\sqrt{1+\hat{\Theta}^2}+\rho_3(\cdot)\right) \qquad (4.26)
\end{aligned}
$$

makes (4.23) satisfy

$$
\dot{V}_2 \leq -x_1^2 - (x_2+x_1\beta_1(\cdot))^2 \leq 0 \qquad (4.27)
$$

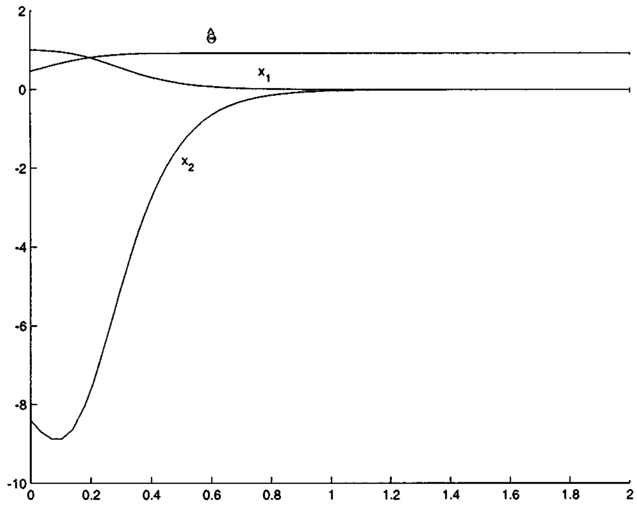

Fig. 1. Transient response of the closed-loop system (4.23)–(4.26), with $x_1(0) = x_2(0) = 1$, $\hat{\Theta}(0) = 0$. True values of parameters—$\theta_1 = 1$ and $\theta_2 = \sqrt{2}$.

where

$$\rho_1(\cdot) = \frac{5}{3}\left|\frac{\partial x_2^*}{\partial x_1}\right| + \left(\frac{5}{3}\frac{\partial x_2^*}{\partial x_1}\beta_1(\cdot)\right)^2$$
$$\rho_2(\cdot) = 1 + \frac{1+\hat{\Theta}^2}{2}\left(\beta_1(\cdot) + \left|\frac{\partial x_2^*}{\partial x_1}\right|\left(1+\frac{1}{3}x_1^2\right)e^{x_1^2/4}\right)^2$$
$$\rho_3(\cdot) = \left(x_1^2\left(1+\frac{1}{3}x_1^2\right)e^{x_1^2/4} + \frac{x_1^2}{2+2\hat{\Theta}^2}\right)^2(3+x_1^2)^2e^{x_1^2/2} + \rho_2(\cdot)\sqrt{1+x_1^2}\sqrt{1+(x_2-x_2^*)^2}(3+x_1^2)e^{x_1^2/4}$$

and

$$\left|\frac{\partial x_2^*}{\partial x_1}\right| = \beta_1(\cdot) + \frac{1}{2}x_1^2(7+x_1^2)e^{x_1^2/4}.$$

The simulation result shown in Fig. 1 indicates that the 1-D adaptive controller (4.26) achieves global stability of the closed-loop system as well as asymptotic state regulation, with a satisfactory dynamic performance and a fast convergent speed of $x(t)$.

V. Applications and Discussions

In this section, we use both physical and academic examples to demonstrate, in the presence of nonlinear parameterizations, some interesting applications of the new adaptive control strategies developed so far.

The first example is the mass-spring mechanical system shown in Fig. 2, where a mass m is attached to a wall through a spring and sliding on a horizontal smooth surface, i.e., resistive force caused by friction is assumed to be zero. The mass is driven by an external force u which serves as a control variable. Let y be the displacement from a reference position.

By Newton's law, the equation of motion for the system is given by

$$m\ddot{y} + F_{sp}(y) = u \tag{5.1}$$

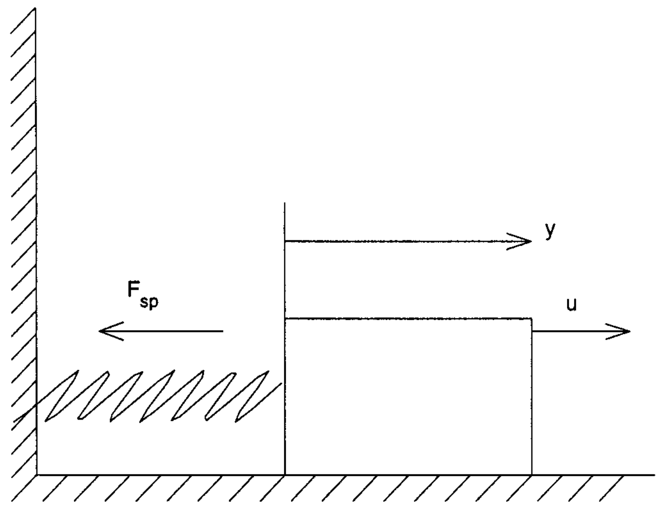

Fig. 2. Mass-spring mechanical system.

where $F_{sp}(y)$ is the restoring force of the spring. Assume that $F_{sp} = F_{sp}(y)$, i.e., is only a function of the displacement and $F_{sp}(0) = 0$. Suppose that we have little knowledge about the spring which may be a linear one or a very complex nonlinear spring with unknown parameters. As discussed in [15], the restoring force of the spring can be modeled as

$$F_{sp}(y) = ky\left(\sum_{i=0}^{q} a_i y^i\right) \tag{5.2}$$

where $a_i's$ and q are unknown parameters. Note that (5.2) represents a family of springs. For example, it becomes a linear spring when $q = 0$ and $a_0 = 1$. In the case when $q = 2$, $a_0 = 1$ and $a_1 = 0$, (5.2) represents a soft spring if $a_2 < 0$ and a hard spring if $a_2 > 0$.

Example 5.1: Consider adaptive control of the mass-spring mechanical system with nonlinear parameterization. We shall show that the problem of adaptive regulation with global stability is solvable, irrespective of the values of q and a_i, $i = 1, \ldots, q$.

To begin with, we define $x_1 = y$ and $x_2 = \dot{y}$ which transform (5.1) into the state-space form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}(u - F_{sp}(x_1)) \tag{5.3}$$

where $m > 0$ can be an unknown mass.

Observe that no matter how big of q is, there exists an unknown constants $\tilde{k} > 0$ and $\theta_1 > 0$, such that

$$|F_{sp}(x_1)| \le \tilde{k}|x_1|(1+|x_1|^q) \le |x_1|e^{x_1^2/2}\theta_1. \tag{5.4}$$

Without loss of generality, in what follows we assume that $1/m \ge 1$.

Obviously, the nonlinearly parameterized system (5.3) satisfies automatically all the conditions of Theorem 4.3 with $n = 2$ and $p_1 = p_2 = 1$. To explicitly design a globally stabilizing smooth adaptive controller, consider $V_1(x_1) = x_1^2/2$ for system (5.3). A direct calculation gives $\dot{V}_1 = -2x_1^2 + x_1\xi_2$, where

$\xi_2 = x_2 + 2x_1$. We then construct $V_2(x_1, x_2) = V_1 + (\xi_2^2/2)$. Clearly,

$$\dot{V}_2 = -2x_1^2 + 2\xi_2^2 - 3x_1\xi_2 + \frac{1}{m}\,\xi_2 u - \frac{1}{m}\,\xi_2 F_{sp}(x_1). \quad (5.5)$$

By (5.4), we have

$$\left|-3x_1\xi_2 - \frac{1}{m}\,\xi_2 F_{sp}(x_1)\right| \le |x_1\xi_2|\left(3 + \frac{1}{m}\,e^{x_1^2/2}\theta_1\right) \le |x_1\xi_2| e^{x_1^2/2}\Theta \quad (5.6)$$

where $\Theta = 3 + (1/m)\theta_1 > 0$ is an unknown constant.

Let $\hat{\Theta}(t)$ be the estimate of $\hat{\Theta}$. Define a positive–definite and proper Lyapunov function

$$U(x_1, x_2, \tilde{\Theta}) = V_2(x_1, x_2) + \frac{\tilde{\Theta}^2}{2}, \qquad \text{with} \quad \tilde{\Theta} = \Theta - \hat{\Theta}.$$

Using (5.5) and (5.6), it is not difficult to show that

$$\dot{U} \le -2x_1^2 + 2\xi_2^2 + \frac{1}{m}\,\xi_2 u + |x_1\xi_2| e^{x_1^2/2}\Theta - \dot{\hat{\Theta}}\tilde{\Theta}. \quad (5.7)$$

By the completion of square

$$\begin{aligned} |x_1\xi_2| e^{x_1^2/2}\Theta &\le \left[\frac{x_1^2}{\sqrt{1+\hat{\Theta}^2}} + \frac{\sqrt{1+\hat{\Theta}^2}\xi_2^2 e^{x_1^2}}{4}\right]\Theta \\ &\le x_1^2 + \frac{\sqrt{1+\hat{\Theta}^2}\xi_2^2 e^{x_1^2}}{4}\,\hat{\Theta} + \Psi(x_1, x_2, \hat{\Theta})\tilde{\Theta} \end{aligned} \quad (5.8)$$

where

$$\Psi(x_1, x_2, \hat{\Theta}) = \frac{x_1^2}{\sqrt{1+\hat{\Theta}^2}} + \frac{\sqrt{1+\hat{\Theta}^2}\xi_2^2 e^{x_1^2}}{4}.$$

Substituting (5.8) into (5.7) yields

$$\dot{U} \le -x_1^2 + \frac{1}{m}\,\xi_2 u + \xi_2^2\left(2 + \frac{\sqrt{1+\hat{\Theta}^2}e^{x_1^2}\hat{\Theta}}{4}\right) + \left(\Psi(x_1, x_2, \hat{\Theta}) - \dot{\hat{\Theta}}\right)\tilde{\Theta}. \quad (5.9)$$

Thus, the smooth adaptive controller

$$\begin{aligned} \dot{\hat{\Theta}} &= \Psi(x_1, x_2, \hat{\Theta}) \\ u &= -\xi_2\left(3 + \frac{\sqrt{1+\hat{\Theta}^2}e^{x_1^2}\hat{\Theta}}{4}\right) \end{aligned} \quad (5.10)$$

is such that $\dot{U} \le -x_1^2 - \xi_2^2$.

The effectiveness of the adaptive controller (5.10) is demonstrated via computer simulation, with the parameters $k = 1$, $m = 1$, $a_1 = a_2 = a_3 = a_4 = 1$, and $q = 4$ in (5.3). The simulation in Fig. 3 indicates that the smooth, 1-D adaptive controller (5.10) does the job, i.e., globally stabilizing the uncertain nonlinear system (5.3) and achieving state regulation, with a good dynamic performance.

The next example is on global adaptive control of a single-link robot with one revolute elastic joint considered, for instance, in [10] and [23].

Example 5.2: A single-link robot with one revolute elastic joint can be, under appropriate conditions, modeled by the nonlinearly parameterized system [23]

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \frac{K}{J_1}\,\zeta_3 - \theta\,\frac{mgl}{J_1}\,\sin\left(\frac{\zeta_1}{\theta}\right) - \frac{K}{J_1}\,\zeta_1 \\ \dot{\zeta}_3 &= \zeta_4 \\ \dot{\zeta}_4 &= \frac{\theta}{J_m}\,u + \frac{K}{J_m}\,(\zeta_1 - \zeta_3) \end{aligned} \quad (5.11)$$

where m, l, K, J_1, J_m, and θ are unknown positive constants.

Global adaptive regulation of system (5.11) was achieved in [23], under the assumptions that all the unknown positive parameters belong to a *known* compact set. However, this crucial condition can be significantly relaxed according to our new adaptive control schemes. As a matter of fact, by Theorem 4.3, the only requirement for achieving global adaptive regulation of (5.11) is that K/J_1 and θ/J_m are bounded below by known positive constants, but their upper bounds need not be known.

The final example is devoted to adaptive control of a nontriangular system with *uncontrollable* linearization.

Example 5.3: Consider the high-order planar system with nonlinear parameterization

$$\begin{aligned} \dot{x}_1 &= x_2^3 + x_2^2|x_1|^\theta \\ \dot{x}_2 &= u \end{aligned} \quad (5.12)$$

where the unknown constant $\theta > 1$.

Clearly, (5.12) is of the form (2.4) but *not in a triangular form*. Observe that by Lemma 2.4,

$$x_2^2|x_1|^\theta \le \tfrac{2}{3}\,|x_2^3| + \tfrac{1}{3}\,|x_1|^{3\theta} \le \tfrac{2}{3}\,|x_2^3| + \tfrac{1}{3}\,|x_1|^3 e^{(1/8)\ln^2(1+x_1^2)}\Theta \quad (5.13)$$

where $\Theta = e^{(9/2)(\theta-1)^2}$. Hence, all the assumptions of Theorem 4.3 or Corollary 4.5 are satisfied. By Theorem 4.3 or Corollary 4.5, there exists a smooth adaptive controller that solves the adaptive stabilization problem for system (5.12).

To design the adaptive controller, consider $V_1(x_1, \tilde{\Theta}) = (1/2)x_1^2 + (1/2)\tilde{\Theta}^2$. Then

$$\dot{V}_1 \le x_1 x_2^3 + \tfrac{2}{3}\,|x_1 x_2^3| + \tfrac{1}{3}\,x_1^4 e^{(1/8)\ln^2(1+x_1^2)}\Theta - \dot{\hat{\Theta}}\tilde{\Theta}.$$

Obviously, the smooth virtual controller

$$\begin{aligned} x_2^* &= -x_1\beta_1(x_1, \hat{\Theta}) \\ \beta_1(x_1, \hat{\Theta}) &:= \left[6 + e^{(1/8)\ln^2(1+x_1^2)}\sqrt{1+\hat{\Theta}^2}\right]^{1/3} \end{aligned}$$

is such that

$$\dot{V}_1 \le -2x_1^4 + \tfrac{5}{3}|x_1|\,|x_2^3 - x_2^{*3}| + \tilde{\Theta}\left(\tfrac{1}{3}\,e^{(1/8)\ln^2(1+x_1^2)}x_1^4 - \dot{\hat{\Theta}}\right).$$

Next, define $\xi_2 = x_2 - x_2^*$ and $V_2(x_1, \xi_2, \tilde{\Theta}) = V_1(x_1, \tilde{\Theta}) + (1/4)\xi_2^4$. A direct calculation gives

$$\dot{V}_2 \le -2x_1^4 + \tfrac{5}{3}\,|x_1|\,|x_2^3 - x_2^{*3}| + \tilde{\Theta}\left(\tfrac{1}{3}\,e^{(1/8)\ln^2(1+x_1^2)}x_1^4 - \dot{\hat{\Theta}}\right) + \xi_2^3\left(u - \frac{\partial x_2^*}{\partial x_1}\,\dot{x}_1 - \frac{\partial x_2^*}{\partial \hat{\Theta}}\,\dot{\hat{\Theta}}\right). \quad (5.14)$$

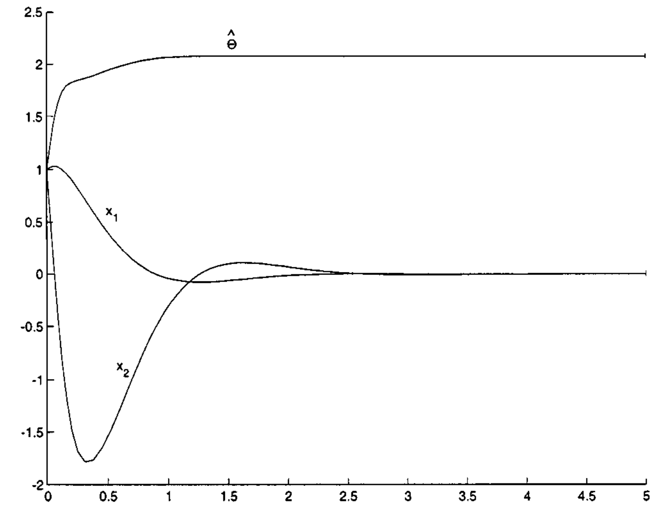

Fig. 3. Adaptive regulation of a mass-spring mechanical system: state trajectories of (5.3)–(5.10) and parameter estimation $\hat{\Theta}$ with $x_1(0) = x_2(0) = \hat{\Theta}(0) = 1$.

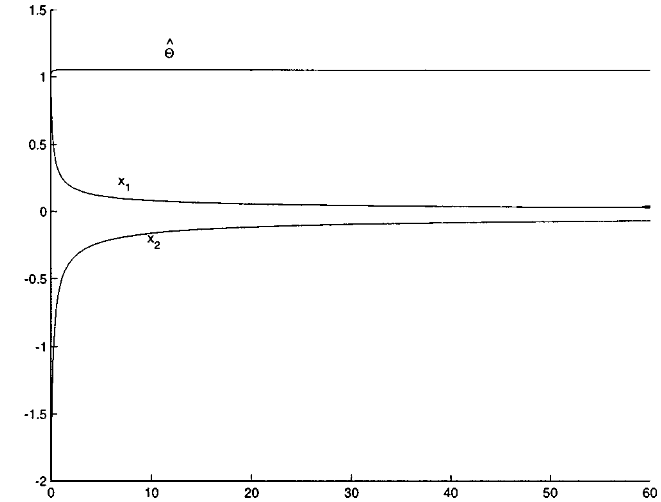

Fig. 4. Transient response of the closed-loop system (5.12)–(5.19) with $x_1(0) = x_2(0) = \hat{\Theta}(0) = 1$, $\theta = 3$.

By (5.13), it is easy to show that

$$\left|\xi_2^3 \frac{\partial x_2^*}{\partial x_1}\dot{x}_1\right| \le \left|\xi_2^3 \frac{\partial x_2^*}{\partial x_1}\right| \left(\frac{5}{3}|x_2^3| + \frac{1}{3}|x_1|^3 e^{(1/8)\ln^2(1+x_1^2)}\Theta\right)$$
$$\le \frac{1}{4}x_1^4 + \xi_2^4\rho_1(\cdot) + \left(\frac{x_1^4}{2+2\hat{\Theta}^2} + \xi_2^4\rho_2(\cdot)\right)\Theta \quad (5.15)$$

where

$$\rho_1(x_1, x_2, \hat{\Theta}) := \frac{5}{3}\left|\frac{\partial x_2^*}{\partial x_1}\right| (\xi_2^2 + 3\xi_2 x_2^* + 3x_2^{*2}) + \frac{3}{4}\left(\frac{5}{3}\left|\frac{\partial x_2^*}{\partial x_1}\right|\sqrt{x_1^4+1}\right)^{4/3}\beta_1^4(\cdot)$$

$$\rho_2(x_1, \hat{\Theta}) := \frac{3}{4}\left(\frac{1+\hat{\Theta}^2}{2}\right)^{1/3} e^{(1/6)\ln^2(1+x_1^2)} \cdot \left(\frac{1}{3}\left|\frac{\partial x_2^*}{\partial x_1}\right|\sqrt{x_1^4+1}\right)^{4/3}$$

are nonnegative smooth functions because

$$\left|\frac{\partial x_2^*}{\partial x_1}\right| = \beta_1(\cdot) + \frac{e^{(1/8)\ln^2(1+x_1^2)}\ln(1+x_1^2)x_1^2\sqrt{1+\hat{\Theta}^2}}{6\beta_1^2(x_1, \hat{\Theta})(1+x_1^2)} > 0$$

is smooth.

Similarly, a direct calculation gives

$$\tfrac{5}{3}|x_1|\,|x_2^3 - x_2^{*3}| \le \tfrac{5}{3}|x_1|\,|x_2 - x_2^*|\left[\tfrac{5}{2}\xi_2^2 + \tfrac{9}{2}x_2^{*2}\right]$$
$$\le \tfrac{1}{4}x_1^4 + \xi_2^4\rho_3(x_1, \hat{\Theta}) \quad (5.16)$$

with

$$\rho_3(x_1, \hat{\Theta}) := \tfrac{3}{8}\left(\tfrac{25}{3}\right)^{4/3} + \tfrac{1}{6}\left(45\beta_1^2(\cdot)\right)^4 > 0.$$

Substituting (5.15) and (5.16) into (5.14), we have

$$\dot{V}_2 \le -\tfrac{5}{4}x_1^4 + \xi_2^4\left(\rho_1(\cdot) + \rho_2(\cdot)\sqrt{1+\hat{\Theta}^2} + \rho_3(\cdot)\right)$$
$$+\left(\tilde{\Theta} + \xi_2^3\frac{\partial x_2^*}{\partial\hat{\Theta}}\right)\left(\Psi_2 - \dot{\hat{\Theta}}\right) - \Psi_2(\cdot)\xi_2^3\frac{\partial x_2^*}{\partial\hat{\Theta}} + \xi_2^3 u \quad (5.17)$$

where

$$\Psi_2 = \frac{1}{3}e^{(1/8)\ln^2(1+x_1^2)}x_1^4 + \frac{x_1^4}{2+2\hat{\Theta}^2} + \xi_2^4\rho_2(\cdot).$$

Finally, it follows from Young's inequality that

$$\Psi_2(\cdot)\xi_2^3\frac{\partial x_2^*}{\partial\hat{\Theta}} \le \Psi_2(\cdot)|\xi_2^3|\,|x_1|\,\frac{e^{(1/8)\ln^2(1+x_1^2)}}{3\beta_1^2(\cdot)}$$
$$\le \tfrac{1}{4}x_1^4 + \xi_2^4\rho_4(\cdot) \quad (5.18)$$

with

$$\rho_4(\cdot) := \frac{3}{4}\left[\left(\frac{1}{3}e^{(1/8)\ln^2(1+x_1^2)} + \frac{1}{2+2\hat{\Theta}^2}\right) \cdot \frac{e^{(1/8)\ln^2(1+x_1^2)}}{3\beta_1^2(\cdot)}\sqrt{x_1^2+1}\right]^{4/3} x_1^4 + \rho_2(\cdot)\xi_2^2\sqrt{\xi_2^2 x_1^2 + 1}\,\frac{e^{(1/8)\ln^2(1+x_1^2)}}{3\beta_1^2(\cdot)}.$$

Putting (5.17) and (5.18) together, it is easy to see that the smooth adaptive controller

$$\dot{\hat{\Theta}} = \frac{1}{3}x_1^4 e^{(1/8)\ln^2(1+x_1^2)}x_1^4 + \frac{x_1^4}{2+2\hat{\Theta}^2} + \xi_2^4\rho_4(x_1, x_2, \hat{\Theta})$$
$$u = -\xi_2\left(1 + \rho_1(\cdot) + \rho_2(\cdot)\sqrt{1+\hat{\Theta}^2} + \rho_3(\cdot) + \rho_4(\cdot)\right) \quad (5.19)$$

yields $\dot{V}_2 \le -x_1^4 - \xi_2^4$, which in turn implies adaptive regulation with global stability.

The simulation result in Fig. 4 shows dynamic performance and parameter estimation of the closed-loop system (5.12)–(5.19). It demonstrates that even in the case of nonlinearly parameterized systems with uncontrollable linearization, global adaptive regulation can be achieved via the new control scheme.

VI. Conclusion

In this paper, we have provided a solution to the problem of adaptive regulation with global stability, for a class of *nonlinearly parameterized* systems with *uncontrollable linearization*.

The systems under consideration are difficult to deal with because they are usually neither feedback linearizable nor affine in the control input. More significantly, they may *not be in a lower triangular* form and involve nonlinear parameterization. The latter has been known as a challenging problem in the field of nonlinear adaptive control.

By using the tool of *adding a power integrator* [19], [20] and coupling it effectively with the new parameter separation technique proposed in Section II, we have shown how a *smooth, one-dimensional* adaptive controller can be explicitly constructed, in a systematic fashion, making the inherently nonlinear systems with nonlinear parameterization global stable with asymptotic state regulation. As a consequence, a solution was obtained to the problem of global adaptive stabilization of feedback linearizable systems with nonlinear parameterization, without imposing any additional condition such as convex or concave parameterization.

Due to the nature of our feedback domination design, it is straightforward to prove that all the adaptive control results obtained in this paper can be directly extended, as shown in Corollary 3.5, to nonlinearly parameterized systems (2.4) with *unknown bounded time-varying signals*, under appropriate conditions such as Assumptions 3.1 and 3.2, or 4.1 and 4.2. In other words, global adaptive regulation is achievable for time-varying nonlinearly parameterized systems such as (2.4) and (3.1), with $\theta = \theta(t)$ and $\theta: \mathbb{R} \to \mathbb{R}^s$ being a continuous function of t, bounded by an *unknown constant*.

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