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# A FREQUENCY DOMAIN PHILOSOPHY FOR NONLINEAR SYSTEMS, WITH APPLICATIONS TO STABILIZATION AND TO ADAPTIVE CONTROL

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Abstract. In this paper we develop, in the nonlinear setting, certain of the basic elements of the frequency domain approach to linear system theory. Thus, we develop analogues of the notion of left and right half plane zeros for systems of relative degree one. It is shown that high gain output feedback stabilizes minimum phase systems of (strong) relative degree one and that lag-lead compensators can also be used to "shape the response" of nonlinear systems. We conclude with a discussion of the performance of high gain adaptive stabilizers for such nonlinear systems.

1. Introduction. In this paper we develop for nonlinear system certain aspects of the standard "frequency domain package" which served as the cornerstone of classical linear system theory. To this end, in Section 2, we present - using the nonlinear geometric control theory - definitions of left half and right half plane zeros and of minimum phase systems. Certainly, the geometric definitions of zero dynamics and of infinite zeros were partially contained or at least anticipated in the early work on (f,g) - invariant distributions (see [5], [6]). And, the notion of relative degree played an important role in Hirschorn's work [4] on system invertibility. In Section 2, we formulate refinements of the definitions so as to include zero locations in half-planes and prove several fundamental facts about finite and infinite zeros. In Section 3, we use this framework to state and prove several results on feedback stabilization of systems. We emphasize the fact that we use classical feedback laws, designed on zero location data, to globally stabilize nonlinear systems. This, we feel, illustrates the power of "the frequency domain package." In Section 4, we indicate how this same approach can be used to design adaptive controllers which universally stabilize "minimum phase" systems having "relative degree one."

2. Zero Dynamics and Infinite Zeros. As a first step in the development of a "frequency domain package" for nonlinear systems, we formulate several definitions which are the nonlinear analogues of the linear notions of left or right half plane zeros and of zeros at infinity. For simplicity, these definitions are given in the scalar real analytic case. We consider then real analytic systems evolving on a real analytic manifold, M of dimension n. Thus, in local coordinates, such a system is described by

$$\dot{x} = f(x) + ug(x) \qquad (2.1a)$$

$$y = h(x) \qquad (2.1b)$$

Denoting the Lie derivative of a function F with respect to a vector field V by $L_V F$, we formulate

Definition 2.1. The system (2.1) has a zero at infinity of multiplicity $\nu_\infty$ if

$$L_g h(x) \equiv L_g L_f h(x) \equiv \dots \equiv L_g L_f^{\nu_\infty - 2} h(x) \equiv 0$$

and

$$L_g L_f^{\nu_\infty - 1} h(x) \not\equiv 0 .$$

This concept, of course, has antecedents in the literature; for example, $\nu_\infty$ plays a central role in Hirschorn's work [4] on invertibility of nonlinear systems, where it was called the relative order of (2.1). We shall also call $\nu_\infty$ the relative degree of (2.1).

Definition 2.2. The system (2.1) has strong relative degree $\nu$ provided it has a zero at infinity of multiplicity $\nu_\infty$ and $L_g L_f^{\nu_\infty - 1} h$ never vanishes.

Turning to the multiplicity of "finite zeros", denote by $\Delta^*$ the maximal locally (f,g)-invariant distribution contained in ker(dh) (see [5], [6] or [8]).

Definition 2.3. The system (2.1) has finite zero dynamics of order $\nu_f$ provided

$$\nu_f = \dim \Delta^* ,$$

where dimension is understood in the generic sense.

Our first result generalizes a result well known in linear systems theory.

Proposition 2.1. Assume the system (2.1) has finite relative degree. Then,

$$\nu_\infty + \nu_f = n .$$

Proof. Following Isidori et al [6], one knows

$$(\Delta^*)^\perp = \sum_{i=0}^{\nu_\infty - 1} d(L_g L_f^i h)$$

and therefore

$$\dim(\Delta^*)^\perp = \nu_\infty . \qquad (2.2)$$

By definition,

$$\operatorname{codim}(\Delta^*)^\perp = \nu_f \qquad (2.3)$$

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and, combining (2.2)-(2.3), we deduce

$$\nu_f + \nu_\infty = \dim M = n \,. \qquad \text{Q.E.D.}$$

We now illustrate $\nu_f$ as the order of a dynamical system, the "zero dynamics" in the case, $\nu_\infty = 1$.

Example 2.1. (A local form for systems of relative degree 1.) To say $\nu_\infty = 1$ is to say there exists $x_0 \in M$ such that $L_g h(x_0) \neq 0$. In particular, $\Delta^* = \ker(dh)$. Thus, there exists a coordinate chart $(x_1,\dots,x_n)$, centered at $x_0$ and defined on a neighborhood U of $x_0$, such that

(i) $\Delta^* + \text{span}(g) = T_x(U)$ , $x \in U$ ;

(ii) $\Delta^* = \text{span}\ \{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{n-1}}\}$ ;

(iii) $\text{span}\ \{g\} = \text{span}\ \{\frac{\partial}{\partial x_n}\}$ .

In these coordinates, setting $z = \frac{\partial}{\partial x_n}$, (2.1) takes the form

$$\dot{z} = f_1(z,x_n)$$

$$\dot{x}_n = f_2(z,x_n) + ug_2(z,x_n)$$

$$y = h(x_n)$$

In the light of the third equation, the second equation may be replaced by

$$\dot{y} = f_2(z,y) + ug_2(z,y)$$

where, of course, $f_2 = L_f h$ and $g_2 = L_g h$. Therefore, (2.1) can be expressed as

$$\dot{z} = f_1(z,y) \qquad (2.4)$$

$$\dot{y} = L_f h(z,y) + uL_g h(z,y) \,.$$

In this setting, the zero dynamics is the (n-1)-th order system

$$\dot{z} = f_1(z,0) \,. \qquad (2.5)$$

Remark. From the local canonical form (2.4) we can derive at least a local form of Hirschorn's Invertibility Theorem [4] for systems of relative degee 1. Explicitly, given an initial condition $x_0$ satisfying $L_g h(x_0) \neq 0$ and a desired output function y(t), one finds a control function u(t) giving rise to y(t) by first integrating (2.4a) for z and then solving (2.4b) for u.

With sufficient regularity hypotheses on $\Delta^*$ and (2.1), one can also derive local forms for systems of higher relative degree. This topic will be treated in more detail in a further paper. Here we will concentrate instead on developing more of the "frequency domain package" in the case, $\nu_\infty = 1$.

Suppose that $x_0 \in M$ is an equilibrium point for (2.1) and that $L_g h(x) \neq 0$ for $x \in h^{-1}(x_0)$. Thus, $h(x_0)$ is a regular value for h and we denote by $L(x_0)$ the analytic submanifold $h^{-1}(h(x_0))$. Note that the zero dynamics (2.5) describes the constrained dynamics

$$\dot{x} = f(x) \quad , \quad h(x) = h(x_0) \qquad (2.6)$$

on the manifold $L(x_0) \cup$. Thus, we refer to (2.6) as the zero dynamics on M. Denote by $W^s(x_0)$ and $W^u(x_0)$ the stable and unstable manifolds of $x_0$ for the system (2.6) and let $W^c(x_0)$ be a center manifold for (2.6). Setting

$$s = \dim W^s(x_0) \ , \ u = \dim W^s(x_0) \ , \ c = \dim W^c(x_0)$$

we will say that (2.1) has s left half plane zeros, u right half plane zeros, and c purely imaginary zeros, in analogy with the linear case. Note that

$$s + u + c = \nu_f \qquad (2.6)$$

We now make a definition which will be essential for the stabilization results which follow.

Definition 2.4. The system (2.1) is minimum phase on M, provided (2.1) has $\nu_f$ left half plane zeros. The system (2.1) is globally minimum phase on M provided it is minimum phase and the zero dynamics (2.6) is globally asymptotically stable.

Generalizing the local form (2.4) for minimum phase systems of relative degree one, we have a globally valid form for globally minimum phase systems having strong relative degree one; viz. there is a diffeomorphism $F\colon M \to R^{n-1} \times R$,

$$F(x) = (z,y) \ , \ z \in R^{n-1} \ , \ y \in R$$

such that (2.1) takes the form

$$\dot{z} = f_1(z,y) \qquad (2.7a)$$

$$\dot{y} = f_2(z,y) + ug_0(z,y) \qquad (2.7b)$$

where

$$y = h(x) \ , \ F^*f_2 = L_f h \ , \ F^*g_0 = L_g h \qquad (2.7c)$$

Since $\Delta^* = \ker dh$ is a constant rank, codimension 1 distribution satisfying

$$\Delta^*(x) + \text{span}\{g(x)\} = T_x M \qquad (2.8)$$

the existence of the form (2.7) follows from the existence of

$$\beta\colon M \to R-\{0\}$$

and a commuting frame $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\}$ for $\Delta^*$ such that

$$[g\beta, \frac{\partial}{\beta x_i}] = 0 \qquad i = 1,\dots, n-1 \,.$$

We note that a frame $\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\}$ exists for $TL(x_0)$ since $L(x_0) = R^{n-1}$, by Milnor's Theorem.

In the light of the existence of the local forms (2.1) and the regularity condition (2.8), the global existence of a frame for $\Delta^*$ and $\beta$ follows, as in the global theory of (f,g)-invariant distributions (see [2], [3]), from the assertions

(i) For all $x \in M$, the leaf $L(x)$ of $\Delta^*$ is simply-connected;

(ii) The map $h: M \to R$ admits local cross-sections

Assertion (ii) follows from the implicit function theorem since $L_g h$ never vanishes on M. If

$$h: \quad M \to R \tag{2.9}$$

were a fiber bundle, then each leaf $L(x)$ of kerdh would be diffeomorphic to $h^{-1}(h(x_0)) = L(x_0) \simeq R^{n-1}$ and hence simply-connected. More generally, a slight modification of the proof of ([1], Proposition 16.3) shows in our situation that

$$\pi_i(L(x)) \simeq \pi_i(L(x_0)) \qquad i \geq 1$$

provided (2.9) is a Serre Fibration. In fact, it suffices to prove:

<u>Proposition 2.2</u>. For globally minimum phase systems with strong relative degree one, the output map

$$h: \quad M \to R$$

has the covering homotopy property with respect to paths.

<u>Proof</u>. We must show that each homotopy

$$\Phi: \quad I\times I \to R$$

admits a lifting

$$\tilde{\Phi}: \quad I\times I \to M \;, \; h\circ\tilde{\Phi} = \Phi \;.$$

System-theoretically, the pathlifting problem is whether a prescribed output function

$$\Phi(0,\cdot) = y: \quad I \to R$$

can be realized as the output of (2.1) with arbitrary initial condition x, subject to the constraint $h(x) = y(0)$. According to Hirschorn's Invertibility Theorem, $y(t)$ can be so realized provided $L_g h(x) \neq 0$. Moreover, since an appropriate input $u(t)$ can be generated by a fixed inverse system driven by $y(t)$ any smooth homotopy $y(s,t)$ can be lifted to a smooth 1-parameter family $x(s,t)$ of trajectories realizing $y(s,t)$. Q.E.D.

We close this section by remarking, conversely, that Hirschorn's Invertibility Theorem follows for such systems from (2.7).

## 3. Stabilization of Globally Minimum Phase Systems

In this section we shall derive results concerning the stabilization of minimum phase, nonlinear systems by constant gain output feedback and by classical linear lag-lead compensation.

To begin, we shall assume (without loss of generality) that the equilibrium point $x_0$ satisfies $h(x_0) = 0$. Next, we assume

(H1) $L_g h > 0$ on all of M;

(H2) Either $L_g h$ is constant or

$$\frac{\partial L_g h}{\partial y}(x) = 0 \;\; = h(x) = 0 \;.$$

Thus, in dimension 1, $L_g h(y)$ is bounded from below by a positive constant.

We can now state one of our principal results:

<u>Theorem 3.1.</u> Suppose the system (3.1) on $R^n$ is globally minimum phase and satisfies (H1)-(H2). Consider the output feedback law

$$u = -ky \tag{3.1}$$

For any bounded open set $U$ $R^n$ there exists $k_U$ such that for all $k > k_U$ the closed-loop system (2.1)-(3.1) is locally and globally asymptotically stable to $x_0$ on U.

<u>Proof.</u> In the coordinates (2.7), the closed-loop system takes the form

$$\dot{z} = f_1(z,y) \tag{3.1a}$$

$$\dot{y} = L_f h(z,y) - kyL_g h(z,y) \tag{3.1b}$$

Setting $\varepsilon = \frac{1}{k}$, we study the asymptotic properties of (3.1) with initial data $(z(o),y(o))$ via singular perturbation theory (see, e.g. [7]). Explicitly, setting $\varepsilon = 0$ we obtain the constraints

$$y\, L_g h(z,y) = 0$$

which implies, by hypothesis, $y = 0$. This leads to the "reduced equation"

$$\dot{\bar{z}} = f(\bar{z},o) \quad , \quad \bar{z}(o) = z(o) \tag{3.2a}$$

and the "boundary layer equation"

$$\dot{\hat{y}} = -k\hat{y}L_g h(z_0,\hat{y}) \;,\; \hat{y}(o) = y(o) \tag{3.2b}$$

By hypothesis, $\frac{\partial}{\partial \hat{y}}(\hat{y}L_g h(z_0,\hat{y})$ is bounded from below, so that Tychonov's Theorem applies. Thus, for fixed initial data $(z_0,y_0)$

$$z_t = \bar{z}_t + O(\tfrac{1}{k})$$

$$y_t = \hat{y}_t + O(\tfrac{1}{k}) \;.$$

Since (3.2b) is globally asymptotically stable in $\hat{y}$ and since (3.2a) coincides with the zero dynamics of a globally minimum phase system, there exists $k = k(z(0), y(0))$ such that

$$\lim_{t\to\infty} z_t = \lim_{t\to\infty} \bar{z}_t = z_0 \quad (3.3a)$$

$$\lim_{t\to\infty} y_t = \lim_{t\to\infty} \hat{y}_t = 0 . \quad (3.3b)$$

Since there exists $k >> 0$ such that (2.1)-(3.1) is locally asymptotically stable, there exists $k'$ depending on $(z(0),y(0))$ such that (3.3) holds for each $k > k'$ for an open neighborhood of $(z(0),y(0))$. If $U$ is any bounded neighborhood of $(z_0,0)$, a standard compactness argument yields the existence of $k >> 0$ such that (2.1)-(3.1) is locally and globally stable on $U$. Q.E.D.

In order to use lag-lead compensation to stabilize "minimum phase" nonlinear systems of higher relative degree it suffices, by induction, to analyze the effect of adding to (2.1) a "pole" which lies sufficiently far to the left of the imaginary axis. The key inductive result in this direction is:

Theorem 3.2. Suppose $\nu_\infty = 1$ and that for some $k$ the closed-loop system (2.1)-(3.1) is locally and globally asymptotically stable to $x_0$ on $R^n$. If $U$ is any bounded neighborhood of $x_0$ there exists a positive $\varepsilon_U$, sufficiently small, so that the closed-loop system with compensator

$$\hat{u}(s) = -\frac{k}{1+\varepsilon s}\hat{y}(s) \quad (3.4)$$

is locally and globally asymptotically stable on $U$.

Sketch of Proof: By hypothesis, the system

$$\dot{x} = f(x) - g(x)kh(x) \quad (3.5)$$

is stable. Since the closed-loop system (2.1)-(3.4) takes the form

$$\dot{x} = f(x) + g(x)u$$

$$\varepsilon\dot{u} = -u + ky$$

the "reduced equation" is (3.5) while the boundary layer equation is simply

$$\dot{\hat{u}} = -\hat{u} + kh(x_0)$$

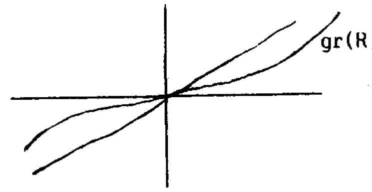
which is linear and stable. As before, the desired result now follows from Tychonov's Theorem. Q.E.D.

## 4. Adaptive Stabilization of Minimum Phase Nonlinear Systems.

In this section, we indicate some adaptive extensions of the "classical" feedback schemes presented in the previous section. As in the linear case, for minimum phase systems having strong linear case the essential stability analysis already appears in the one dimensional case. For this reason, we analyze this case in more detail. Consider therefore the system evolving on $R$:

$$\begin{aligned} \dot{x} &= f(x) + ug(x) , & f(0) &= 0 \\ y &= h(x) , & h(0) &= 0 \end{aligned} \quad (4.1)$$

and suppose the ratio $R = L_fh/L_gh$ satisfies a sector condition:




We omit the proof of the following result which is straightforward.

Theorem 4.1. Suppose (4.1) satisfies $L_gh > 0$ and that $R$ satisfies a sector condition. Consider the adaptive stabilization scheme:

$$\dot{k} = y^2 \quad , \quad u = -ky . \quad (4.2)$$

Then, the closed-loop system (4.1)-(4.2) satisfies, for all initial data, the conditions

(i) $k_t$ remains bounded as $t \to \infty$ ;

(ii) $x_t \to 0$ , $t \to \infty$ .

Remarks. 1. The conclusion in Theorem 4.1 remains valid for the system

$$\dot{x} = x^2 + u \quad , \quad y = x \quad (4.3)$$

which does not satisfy a sector condition.

2. Simulations show that the conclusion in Theorem 4.1 remains valid only for small initial data ($x_0 < 2^{-1/2}$) for the system

$$\dot{x} = x^3 + u \quad , \quad y = x . \quad (4.4)$$

The intuition behind Remarks 1 and 2 is simply that, according to Theorem 3.1, for any $x_0$ there exists $k >> 0$ such $u = -kx$ stabilizes $x_0$, but no fixed $k$ stabilizes all initial data. For (4.3), the control law (4.2) allows $k$ to grow sufficiently fast with respect to the excursions in $x_t$, while the growth is not fast enough for (4.4). We propose, then, the following adaptive control scheme:

Theorem 4.2. Suppose (4.1) satisfies $L_gh > 0$ where $f$, $g$, and $h$ are polynomials. Consider the control law

$$\dot{k} = e^{y^2} \quad , \quad u = -ky \quad (4.5)$$

Then (4.1)-(4.5) satisfies, for all initial data $(x_0,k_0)$

(i) $k_t$ remains bounded as $t \to \infty$;

(ii) $x_t \to 0$ as $t \to \infty$ .

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